

T H È S E

présentée à

L'UNIVERSITE DE PARIS XI
CENTRE D'ORSAY

pour obtenir le grade de

DOCTEUR ES SCIENCES PHYSIQUES

par

André NEVEU

Renormalisation et Nombres Quantiques Physiques dans les Modèles Duaux

Soutenue le 18 Juin 1971 devant la Commission d'Examen

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LISTE DES ARTICLES PRESENTES

- 1 - Renormalized Self-Energy Operator in the Dual Resonance Model
- 2 - The Primitive Graphs of Dual Resonance Models
- 3 - Divergences and Renormalization in Multiloop Dual Diagrams
- 4 - Tachyon-Free Dual Model with a Positive-Intercept Trajectory
- 5 - Factorizable Dual Model of Pions
- 6 - Reformulation of the Dual Pion Model
- 7 - Quark Model of Dual Pions.

INTRODUCTION

Dans les trois années qui se sont écoulées depuis l'article initial de Veneziano, beaucoup d'efforts ont été consacrés à la construction de modèles duaux qui aient quelque ressemblance avec la réalité, ou qui puissent servir de base à un modèle entièrement soluble de théorie des hadrons. Un grand progrès dans cette dernière direction a été fait par la suggestion de Kikkawa, Sakita et Virasoro de considérer la généralisation à la fonction à N points de l'amplitude de Veneziano comme un terme de Born analogue aux diagrammes en arbre d'une théorie lagrangienne des champs. On peut ainsi construire des diagrammes de Feynman compatibles avec la dualité. Quand on impose la factorisation sur les lignes internes de ces diagrammes, on obtient un résultat qui diverge exponentiellement. La première partie de cette thèse étudie quelques problèmes de la renormalisation de ces divergences.

Une méthode de renormalisation de la divergence des diagrammes duaux plans à une boucle est exposée dans la thèse de Joël Scherk. Cette méthode est fort semblable à une renormalisation de masse en théorie quantique des champs, et est compatible avec la dualité, l'unitarité, et le comportement à la Regge. Afin d'étendre ces résultats aux diagrammes à plusieurs boucles, il faut montrer qu'elle est compatible avec la factorisation. Le premier article étudie cette compatibilité en construisant dans le formalisme opératoire l'opérateur de self-énergie plan à une boucle et en le renormalisant.

Les divergences des diagrammes plans à N boucles ont été étudiées par Joël Scherk et Michio Kaku. Ils ont montré que le nombre

de ces divergences croît comme $2^N - 1$ avec le nombre de boucles, alors que le contre-terme défini pour une boucle dans le premier article n'en compense que N . En se restreignant d'abord au cas de deux boucles, le deuxième article montre que l'apparition de la divergence supplémentaire est liée aux propriétés topologiques du diagramme, et que cette divergence peut être mise en évidence par une transformation de dualité, et reliée à la divergence qui apparaît dans la self-énergie non plane à une boucle. Cette transformation et les propriétés de cette self-énergie permettent alors de compenser la nouvelle divergence par l'introduction d'un contre-terme supplémentaire. Ce contre-terme a la même interprétation physique d'une renormalisation de masse que le contre-terme pour une boucle, mais d'un ordre supérieur dans la constante de couplage. On conserve ainsi au niveau de deux boucles la renormalisabilité des diagrammes plans mise en évidence au niveau d'une boucle, et son interprétation physique d'une renormalisation de masse. L'article étudie ensuite plus généralement les propriétés des diagrammes à N boucles. A l'aide d'arguments topologiques simples, il montre que des $2^N - 1$ divergences d'un diagramme à N boucles, une seule est présente, qui ne l'était pas pour les diagrammes à $N - 1$ boucles. Ceci permet de définir un processus itératif de renormalisation de ces divergences : on compense d'abord la nouvelle divergence apparaissant à l'ordre N par l'introduction d'un nouveau contre-terme, d'ordre $2N$ dans la constante de couplage, puis on renormalise les autres divergences à l'aide des contre-termes définis aux ordres inférieurs. La propriété fondamentale de cette méthode est de conserver à chaque ordre dans la constante de couplage la possibilité d'interpréter les contre-termes comme étant des renormalisations de masse.

Tout diagramme dual plan peut par dualité être réduit à l'itération du tadpole plan à une boucle. Ceci simplifie grandement la construction et l'étude des diagrammes plans en général. Il est donc intéressant de chercher une semblable propriété de réduction pour les diagrammes non plans. Kikkawa, Klein, Sakita et Virasoro ont montré que l'énumération des diagrammes duaux est équivalente à la classification des variétés compactes à deux dimensions à bords. En utilisant dans le troisième article les propriétés de ces variétés, on montre que le diagramme dual le plus général peut être construit en attachant à un diagramme en arbre une ou plusieurs fois quatre graphes fondamentaux : le tadpole plan à une boucle, le tadpole non orientable à une boucle, la self-énergie non plane à une boucle et le tadpole orientable non plan à deux boucles. Cette réduction simplifie grandement l'étude des divergences et de la renormalisabilité éventuelle d'une théorie duale.

Malgré les progrès dans notre compréhension des propriétés mathématiques du modèle dual auxquels la première partie de cette thèse a pu contribuer, un des principaux défauts de ce modèle est son manque de ressemblance avec le monde physique des hadrons. Les premières tentatives en vue d'introduire des nombres quantiques plus réalistes ajoutaient au modèle une superstructure décrivant le spin et les nombres quantiques internes des quarks. Ces tentatives ont rencontré des difficultés insurmontables dues aux 'ghosts' et aux doublets de parité. L'introduction de nouveaux opérateurs s'est révélée plus fructueuse : la deuxième partie de cette thèse est consacrée à la construction de nouveaux modèles suivant cette idée.

Parmi les problèmes de la construction d'un modèle dual pour les mésons physiques se trouve le fait que la trajectoire principale $(\rho - f^0)$ commence naturellement par une particule de spin 1 (ρ) et non une particule de spin zéro, et que cette trajectoire de G-parité positive est une demi-unité au-dessus de la trajectoire du π , de G-parité négative, alors que dans le modèle dual ordinaire la trajectoire principale commence toujours par une particule de spin zéro, et que l'espacement des trajectoires est un nombre entier. Dans un premier article, est introduit un nouvel ensemble d'opérateurs permettant de construire un modèle dual ayant deux sortes de trajectoires : elles sont séparées par une demi-unité, et leurs G-parités sont différentes. L'état fondamental de la trajectoire principale a un spin un. Malheureusement, la G-parité négative de cet état ne permet pas de le considérer comme un candidat raisonnable pour le méson ρ . Un autre inconvénient du modèle décrit dans cet article est que l'introduction de nouveaux opérateurs vectoriels augmente le nombre de 'ghosts' de la théorie, que l'on ne peut plus espérer compenser tous avec les seules conditions subsidiaires de Virasoro. C'est dans le deuxième article que fut découverte une légère mais cruciale modification qui permet une grande amélioration des propriétés du modèle : le nouveau modèle possède des mésons π et ρ ayant les bons nombres quantiques. La trajectoire principale $(\rho - f_0)$ commence bien avec un état de spin un (ρ) ; la trajectoire du π est située une demi-unité en-dessous de celle du ρ ; des trajectoires avec des couplages de parité anormale aux pions $(\omega - A_2, \eta')$ apparaissent de façon naturelle ; tous ces états $(\pi, \rho, f^0, \omega, A_2, \eta')$ ont univoquement les G-parités et isospins des états expérimentalement observés ; l'amplitude $\pi\pi \rightarrow \pi\pi$ coïncide avec la suggestion initiale de Veneziano, et

l'amplitude $\pi\pi \rightarrow \pi\pi$ avec celle de Shapiro et Lovelace. Le spectre des premiers états de ce modèle ne comporte aucune particule sans relation avec une particule du spectre expérimental des mésons non étranges (ou d'un modèle de quarks non relativiste). Ce spectre, aussi loin qu'il ait été exploré, ne contient que des états de norme positive. On ne saurait cependant dire que ce modèle de pions décrit convenablement le monde physique, car il a plusieurs défauts : les trajectoires ayant un couplage de parité normale aux pions (π , ρ , f_0) sont trop hautes d'une demi-unité (ce qui signifie $m_\pi^2 = -1/2$ et $m_\rho^2 = 0$). En revanche, les trajectoires à couplage de parité anormale (ω , A_2 , η') sont au bon endroit ($m_\omega^2 = 1/2$, $m_{\eta'}^2 = 1$). A cause de cette non-dégénérescence du ω et du ρ , une généralisation triviale à $SU(3)$ n'est guère satisfaisante, car elle impliquerait l'existence de deux octets de mésons vectoriels, l'un portant les couplages de parité normale, l'autre les couplages de parité anormale. Nous considérons que la propriété la plus remarquable du modèle est de contenir de nouveaux opérateurs de jauge en plus des opérateurs de jauge introduits par Virasoro dans le modèle ordinaire. Ces nouveaux opérateurs sont à l'origine de la plupart des remarquables propriétés physiques du modèle. Il est probable qu'une meilleure compréhension de l'algèbre qu'ils engendrent soit d'une grande importance pour la construction de modèles encore plus réalistes.

Toutes les propriétés énumérées ci-dessus furent d'abord prouvées à partir de la conjecture que l'état fondamental est parasite, c'est-à-dire se découple d'un nombre quelconque de pions. Ce découplage a été vérifié par calcul explicite pour quatre et six pions, mais une preuve générale et plus élégante était nécessaire. Ceci est l'objet du

troisième article ; cette conjecture y est prouvée par des méthodes algébriques simples. Ces méthodes permettent de reformuler naturellement le modèle dans un nouvel espace de Hilbert. Cet espace a l'avantage de contenir moins d'états parasites, et donc de permettre une étude plus aisée du spectre des états physiques.

Le quatrième et dernier article propose un champ de recherches nouveau en introduisant pour la première fois des fermions dans un modèle dual. Ramond a récemment proposé un intéressant principe de correspondance reliant le spectre du modèle dual ordinaire et l'équation de Klein-Gordon pour une particule scalaire. Par ce même principe de correspondance, à partir de l'équation de Dirac, il a proposé une équation d'onde pour des fermions duaux. Les solutions de cette équation sont situées sur des trajectoires linéaires. L'article introduit une interaction dans ce modèle de fermions en construisant un vertex pour l'émission d'un pion pseudoscalaire à partir d'une ligne de fermions. Maintenir la validité des jauges de la ligne de fermions en présence de l'interaction impose la condition $m_{\pi}^2 = -1/2$, ce qui est la même valeur que celle du modèle précédemment décrit. En factorisant sur le premier pôle du canal fermion-antifermion, on obtient effectivement l'amplitude à N pions décrite dans les deux précédents articles. A partir de ce résultat, de nombreuses directions de recherche sont possibles : tout d'abord il devrait être possible de généraliser la factorisation afin de décrire le couplage d'un état mésonique arbitraire au fermion. Il est suggéré également de considérer les fermions comme étant plutôt des quarks que des embryons de baryons ; il faudrait alors considérer la diffusion d'un nombre quelconque de ces quarks, et en particulier, on pourrait essayer

de construire des baryons en factorisant dans un canal à trois quarks. Avoir un tel modèle de quarks peut se révéler fort utile pour la construction de modèles duaux ayant de plus en plus de propriétés physiques intéressantes.

RENORMALIZED SELF-ENERGY OPERATOR IN THE DUAL-RESONANCE MODEL*

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A divergent planar self-energy operator and a real dual counterterm operator that cancels the divergence are obtained. These operators can be used to calculate multiloop planar diagrams.

The dual-resonance model that has grown out of the Veneziano formula holds promise for providing the basis for an eventual formulation of a unitary Regge-behaved theory of hadrons. Before this goal can be achieved, however, many obstacles must still be overcome. A potentially very serious difficulty is the exponential divergence of planar one-loop amplitudes. Two of us [1] have recently shown how to construct a real dual counterterm that cancels this divergence. If this work is to form the basis of a consistent procedure it is necessary for the counterterm to have appropriate factorization properties. In this paper we make the factorization manifest by first obtaining a divergent self-energy operator with which the planar one-loop diagrams can be recovered and then constructing a real dual counterterm operator that cancels divergence. Multiloop diagrams can then be calculated.

We begin with a deceptively simple, but very useful theorem. It enables us to extend the domain of applicability of various operators from that for which they were originally obtained to the complete Hilbert space. Suppose $\hat{O}_1$ and $\hat{O}_2$ are two operators, associated with momentum p (such as propagators, for example) acting in the space of harmonic-oscillator states of the dual-resonance model [2]. Suppose further that $\hat{O}_1$ and $\hat{O}_2$ have the same matrix elements between any pair of states corresponding to external trees in multiperipheral configurations. The operator identity

$$\hat{O} = P^+(-p)(\hat{O}_1 - \hat{O}_2)P(p) = 0 \quad (1)$$

then holds, where $P(p)$ is the projection operator onto the space of physical states of momentum p defined in ref. [3]. Namely,

$$P(p) = -(A^+(-p) - \alpha_0) [A(p)(A^+(-p) - \alpha_0)]^{-1} A(p) + I \quad (2)$$

where

$$A(p) = \sum_{n=1}^{\infty} (na_n^+ a_n - \sqrt{n(n+1)} a_n^+ a_{n+1}) + p \cdot a_1 - \frac{1}{2} p^2.$$

The proof of this theorem depends entirely on the assumption that the only Ward identities giving rise to spurious states are those identified by Fubini and Veneziano [4]† and incorporated in the operator P . Using this assumption (which in fact we believe can be proved), one can see that eq. (1) is true. If $|t_1\rangle$ and $|t_2\rangle$ are multiperipheral tree states, then $\hat{O}|t_1\rangle$ is orthogonal to any $|t_2\rangle$. Thus the assumption of no new Ward identities implies that there exists a state $|s\rangle$, not necessarily a tree, such that

$$\hat{O}|t_1\rangle = A^+|s\rangle.$$

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** Procter Fellow.

*** NATO Fellow.

† For the special intercept choice $\alpha_0 = 1$ there are, in fact, additional Ward identities [5].

It then follows that $\bar{O}$ must in fact annihilate multiperipheral trees,

$$\bar{O}|t_1\rangle = P^+ \bar{O}|t_1\rangle = P^+ A^+|s\rangle = 0.$$

Thus, again assuming no new Ward identities, there must be an operator G such that $\bar{O} = GA$. We then conclude that $\bar{O}$ vanishes because

$$\dot{\bar{O}} = \bar{O}P = GAP = 0,$$

which proves the theorem.

This theorem allows one to prove a number of results that are very difficult to obtain by direct algebraic manipulations. For example, the vertex operator of Sciuto [6], derived so as to be valid between multiperipheral tree configurations, is neither symmetric in the three particles nor fully factorized. However, a fully symmetric and factorized vertex operator has been found by Caneschi et al. [7]. As it differs from Sciuto's only by a twist and a gauge transformation, it is also correct between multiperipheral trees. Our theorem (slightly restated to apply to a vertex configuration) now implies that the operator defined by attaching physical-state projection operators to the symmetric vertex operator is generally valid, without restriction to multiperipheral tree states. The explicit formula is*

$$W(p_1, p_2, p_3) = P_a(p_1) P_b(p_2) P_c(p_3) \exp\{p_3 \cdot a^+ + p_1 \cdot b^+ + p_2 \cdot c^+ + [a^+, c^+]_- + [c^+, b^+]_- + [b^+, a^+]_-\} |0\rangle \quad (3)$$

where

$$[a^+, c^+]_- = \sum_{n,m=1}^{\infty} \frac{a_n^+ \cdot c_m^+}{\sqrt{nm} B(-n, m)}.$$

Similar arguments can be used to prove an operator version of the duality relation. The main use of the theorem in this paper, however, will be to deduce the general form of the planar self-energy operator from the factorization of one-loop diagrams.

The self-energy operator, with propagators attached, is constructed using the operator techniques of Fubini et al. [2] incorporating the vertex operator of Sciuto [6]. This step closely parallels a recent paper of Guérin [8]. Using the notation of, e.g., Caneschi et al. [7] and the conventions of fig. 1,

$$D(R_c, p^2) \Sigma(p) D(R_c, p^2) = g^2 \int_0^1 dx_1 dx_2 dy_1 dy_2 \frac{[(1-x_1)(1-x_2)/x_1 x_2]^{\alpha_0}}{x_1 x_2 (1-x_1)(1-x_2)(1-y_1)(1-y_2)} (y_1 y_2)^{-\alpha(p^2)-1} \int d^4 k X(k, p) \quad (4)$$

where

$$X(k, p) = x_1^{-\frac{1}{2}(k-p)^2} \xi^{-\frac{1}{2}k^2} y_1^{R_c} \text{Tr}_a \{ \exp(-p \cdot a^+ + [a^+, c^+]_- + k \cdot c^+) \exp([a, c^+]_+ - p \cdot a) |O_c\rangle \\ \times \xi^{R_a} \langle O_c | \exp([a^+, c]_+ + p \cdot a^+) \exp(p \cdot a + [a, c]_- - k \cdot c) x_1^{R_a} y_2^{R_a} \rangle$$

and

$$\xi = x_2(1-y_1)(1-y_2).$$

We have defined an operator in the space of c oscillators; the a oscillators, which are identified with the loop, are eliminated in the trace calculation. Taking matrix elements between external coherent states and performing the momentum integrations,

$$\int d^4 k \langle \lambda_r | X | \mu_r \rangle = 4\pi^2 [f(W)]^{-4} (\ln W)^{-2} \exp \Delta \quad (5)$$

where

* We have explicitly checked that this symmetric vertex when iterated and sandwiched between external trees yields n -point functions in non-multiperipheral configurations.

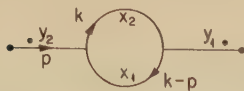


Fig. 1. Diagrammatic representation of the self-energy operator displaying the notation used in the calculation of the text.

$$W = x, \xi, f(w) = \prod_{n=1}^{\infty} (1-w^n),$$

and

$$\Delta = -\frac{p^2}{2} \ln x_1 + \frac{(p \ln x_1 + K_0 - M_0)^2}{2 \ln w} + \sum_{n=1}^{\infty} \frac{1}{n(1-w^n)} [p^2(2w^n - x_1^n - \xi^n) + p \cdot (K_n^+ - M_n^+)(\xi^n - w^n) + P \cdot (K_n^- - M_n^-)(x_1^n - w^n) + x_1^n K_n^- \cdot M_n^- + \xi^n K_n^+ \cdot M_n^+ + w^n (K_n^- \cdot K_n^+ + M_n^- \cdot M_n^+)],$$

$$K_0 = \sum_{r=1}^{\infty} \lambda_r y_1^r / \sqrt{r}, \quad K_n^{\pm} = \sum_{r=1}^{\infty} \lambda_r y_1^r / \sqrt{r} B(\pm n, r),$$

$$M_0 = \sum_{r=1}^{\infty} \mu_r y_2^r / \sqrt{r}, \quad M_n^{\pm} = \sum_{r=1}^{\infty} \mu_r y_2^r / \sqrt{r} B(\pm n, r).$$

As has been observed by others [9], the factor $[f(w)]^{-4}$ causes the remaining integral to be exponentially divergent in the corner where x_1 and x_2 approach 1 and y_1 and y_2 approach 0. Therefore a cutoff factor $\theta(1-w-\epsilon)$ should be inserted with the understanding that we will later be able to let ϵ go to 0.

The next step is to recast the formula for the self-energy operator into a form appropriate for identifying the counterterm. Using the identities

$$K_0 = - \sum_{r=1}^{\infty} \frac{\sqrt{r} \lambda_r y_1^r}{r!} \left(-\frac{\partial}{\partial \rho}\right)^r \ln \rho \Big|_{\rho=1}, \quad K_n^{\pm} = \sum_{r=1}^{\infty} \frac{\sqrt{r} y_1^r \lambda_r}{r!} \left(-\frac{\partial}{\partial \rho}\right)^r (\rho^{\mp n} - 1) \Big|_{\rho=1}, \quad (6)$$

and similar formulas for M_0 and $M_n^{\pm}$, we can rewrite Δ in the form

$$\Delta = p^2 \ln \psi(x_1) - p_1 \sum_{r=1}^{\infty} \frac{(\lambda_r' - \mu_r')}{r!} \left(-\frac{\partial}{\partial \rho}\right)^r \ln \left[\frac{(1-\rho) \psi(\rho x_1)}{\psi(\rho) \psi(x_1)} \right]_{\rho=1} \quad (7)$$

$$- \sum_{r,s=1}^{\infty} \frac{1}{r! s!} \left(-\frac{\partial}{\partial \rho_1}\right)^r \left(-\frac{\partial}{\partial \rho_2}\right)^s \left\{ \lambda_r' \cdot \mu_s' \ln \left[\frac{\psi(x_1) \psi(\rho_1 \rho_2 x_1)}{\psi(\rho_1 x_1) \psi(\rho_2 x_1)} \right] - \frac{\lambda_r' \cdot \lambda_s' + \mu_r' \cdot \mu_s'}{2} \ln \left[\frac{(\rho_1 - \rho_2) \psi(\rho_1) \psi(\rho_2)}{\psi(\rho_2 / \rho_1) (1 - \rho_1) (1 - \rho_2)} \right] \right\}_{\rho_1 = \rho_2 = 1}$$

where we have defined

$$\psi(x) \equiv \psi(x, w) = \exp \left\{ -\frac{\ln x \ln w/x}{2 \ln w} + \sum_{n=1}^{\infty} \frac{2w^n - x^n - (w/x)^n}{n(1-w^n)} \right\},$$

which can be expressed in terms of Jacobi θ functions,

$$\psi(x) = -2\pi i \exp \frac{\ln^2 x}{2 \ln w} \theta_1 \left(\frac{\ln x}{2\pi i} \middle| \frac{\ln w}{2\pi i} \right) / \theta_1' \left(0 \middle| \frac{\ln w}{2\pi i} \right) = -\ln w \theta_1 \left(\frac{\ln x}{\ln w} \middle| \frac{-2\pi i}{\ln w} \right) / \theta_1' \left(0 \middle| \frac{-2\pi i}{\ln w} \right).$$

$\psi(x)$ has the following interesting properties

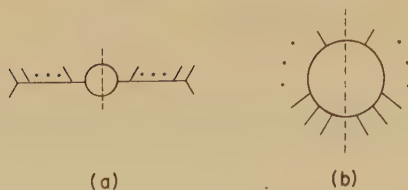


Fig. 2. a) One-loop N -point function obtained by taking a matrix element of the self-energy operator; b) A graph equivalent to the one in a) because of duality.

$$\psi(x) = \psi(w/x) = -\psi(1/x), \quad \lim_{x \rightarrow 1} (1-x)^{-1} \psi(x) = 1.$$

Notice that in eq. (7) we have inserted noncontributing factors into the logarithms in such a way that the sums may be extended to r and $s = 0$.

The usual planar N -point one-loop amplitude [1,9] can be written in the form

$$F_N(p_1, \dots, p_N) = 4\pi^2 g^N \int_0^1 \left(\prod_{i=1}^N dx_i (1-x_i)^{\alpha_0-1} \right) W^{-\alpha_0-1} [f(w)]^{-4} (\ln w)^{-2} \\ \times \prod_{1 \leq i < j \leq N} \psi(x_{i+1} x_{i+2} \dots x_j)^{-p_i \cdot p_j}, \quad (8)$$

where $w = \prod_{i=1}^N x_i$. To recover this formula from our operator expressions one first notes that the deriv-

ative operators in eq. (7) may be eliminated using Taylor's formula when the usual expressions appropriate to multiparticle external states are used for λ_r and μ_r . This amounts to calculating F_N in the configuration of fig. 2a. Next, one need only determine the right change of integration variables to show that the result is identical to eq. (8), which corresponds to the configuration of fig. 2b. Not surprisingly, it turns out that the change of variables is the same as one would use to demonstrate the equality of the tree diagrams on the two sides of the dotted lines in figs. 2a and 2b.

We can now construct a counterterm operator $\tilde{S}(p)$, with propagators attached. One requirement is that when $\epsilon \rightarrow 0$ $D(\Sigma - \tilde{S})D$ has finite matrix elements between multiparticle external states.* $\tilde{S}(p)$ is also required to give a dual symmetric amplitude $\tilde{F}_N$ containing single and double poles, but no cuts. The renormalized amplitude $F_N - \tilde{F}_N$ then has the same absorptive parts as F_N alone, so that perturbative unitarity is still satisfied. As we are subtracting single and double poles only, while preserving factorization properties, the situation is quite similar to the renormalization of one-loop diagrams in quantum field theory. The rule for constructing the counterterm operator is now extremely simple: replace each function $\psi(x)$ appearing in eq. (7) by

$$\tilde{\psi}(x) = -\frac{\ln w}{\pi} \sin\left(\pi \frac{\ln x}{\ln w}\right).$$

When the counterterm operator, so defined, is evaluated between multiparticle external states one finds that $\tilde{F}_N$ is given by eq. (8) with ψ replaced by $\tilde{\psi}$.** The divergence cancels from $F_N - \tilde{F}_N$ (as is discussed in detail in ref. [1]) because

$$\psi(x) - \tilde{\psi}(x) \underset{w \rightarrow 1}{\sim} O\left(\exp \frac{4\pi^2}{\ln w}\right).$$

The dual pole structure of $\tilde{F}_N$ is also discussed in ref. [1].

The counterterm operator defined above is not unique because additional finite counterterms could also be included. One way to construct such a term is to drop the factor $[f(w)]^{-4}$ from the infinite

* Strictly speaking, logarithmic divergences still remain. They reflect the need for a renormalization of the masses of the external lines. This divergence is present in eq. (8) even with the cutoff described in the text.

** This counterterm corresponds to a slight modification of the prescription of ref. [1], which was not properly factorizable.

counterterm. It remains to be seen whether principles that restrict or eliminate this freedom will emerge. Another problem is the presence of spurious states contributing to the internal lines of the loop. Their contribution is removed by inserting a factor of $1-w$ into eq. (4) [3,9].

So far, we have constructed the self-energy operator with external propagators attached. Combining eqs. (4) and (5) gives

$$\langle \lambda_r | D(R, p^2) \Sigma(p) D(R, p^2) | \mu_r \rangle = 4\pi^2 g^2 \int_0^1 \frac{dx_1 dx_2 dy_1 dy_2}{\ln^2 w} (1-y_1)(1-y_2) \times \\ \times [f(w)]^{-4} W^{-\alpha_0-1} [(1-x_1)(1-x_2)(1-y_1)(1-y_2)]^{\alpha_0-1} (1-w)(y_1 y_2)^{-\alpha(p^2)-1} \exp[\Delta(p, x, y, \lambda', \mu')] \quad (9)$$

with $\Delta(p, x, y, \lambda', \mu')$ given by eq. (7) and $w = x_1 x_2 (1-y_1)(1-y_2)$. If we hope to be able to calculate multi-loop diagrams it is clearly necessary to find the operator $\Sigma(p)$ by itself, i.e., to factorize out the propagators from eq. (9). This can be achieved in two steps. The first step is to drop the factors corresponding to the propagators and write

$$4\pi^2 g^2 : \int_0^1 \frac{dx_1 dx_2}{x_1 x_2 \ln^2 w} (1-w) [f(w)]^{-4} W^{-\alpha_0} \exp[\Delta(p, x, y, a^+, a)] : , \quad (10)$$

where the dots indicate normal ordering of the creation and annihilation operators. This result is not yet complete. Through its dependence on w , the expression in eq. (10) still depends on $1-y_1$ and $1-y_2$, which are relics of the discarded propagators.

To complete the construction of the self-energy operator we sandwich expression (10) between projection operators (so that our fundamental theorem will apply) and replace $1-y_1$ and $1-y_2$ by appropriate operators. We recall the identity [3]

$$A(p) D(R, p^2) = D(R+1, p^2) A(p) + L_0(p) (D(R, p^2) - D(R+1, p^2)) \quad (11)$$

where

$$L_0(p) = R - \frac{1}{2} p^2 .$$

Multiplying by a projection operator on the right in eq. (11) eliminates the $A(p)$ term leaving

$$L_0^{-1} A D(R) P = [D(R) - D(R+1)] P . \quad (12)$$

Therefore, the operator $L_0^{-1} A$ "reads off" the variable $1-y$, where y is the integration variable of an adjacent propagator. More generally, $F(L_0^{-1} A)$ is equivalent to $F(1-y)$. We therefore expand the integrand of eq. (10) so that all y_1 's are on the left and all y_2 's are on the right (this supercedes the other normal ordering) and then replace $1-y_2$ by $L_0^{-1}(p) A(p)$ and $1-y_1$ by $A^+(-p) L_0^{-1}(p)$. The resulting expression is clearly equivalent to the one with which we started when inserted between multiperipheral tree states. Our fundamental theorem therefore guarantees that this is the unique planar self-energy operator valid for all purposes including the construction of multiloop diagrams. The normal ordering of the y -variable operators poses no problems, even for multiloop diagrams. These operators can always be immediately replaced by the integration variables of the adjacent propagators.

In conclusion, we have constructed a unique, well-defined, and fully renormalized planar self-energy operator for the version of the dual-resonance model in which spurious states are removed. This operator can be used to calculate multiloop diagrams. The nontrivial problems that arise when one attempts to make fully explicit the two-loop formulas (i.e., evaluate all the operators) involve the diagonalization of the quadratic forms in eq. (7) and the elimination of the projection operator from in between the two self-energy operators.

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THE PRIMITIVE GRAPHS OF DUAL - RESONANCE MODELS*

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We show that the problem of constructing and, if necessary, renormalizing general diagrams in dual-resonance models can be reduced to the study of only four primitive graphs, three of which are tadpoles, the last one being a self-energy graph. We discuss consequences and applications of this reduction.

The process of unitarizing a factorizable dual-resonance model requires constructing Feynman-like diagrams compatible with duality and factorization [1,2]. As was shown in ref. 1, there exist different Feynman-like diagrams which are connected by duality. For instance, any planar diagram can be reduced to the iteration of a single planar tadpole. This reduction simplifies the problem of renormalizing planar diagrams: one has just to renormalize the planar tadpole operator, and to check that the iteration of this renormalized operator does not create new infinities. It is thus interesting to look for a similar reduction of non-planar diagrams.

The set of all Feynman-like diagrams related by duality form an equivalence class, called a duality diagram. In ref. 2 the enumeration of duality diagrams was shown to be equivalent to the classification of compact two-dimensional manifolds with boundaries [e.g. 3]. By using the properties of these manifolds, we show that any duality diagram can be built up by attaching to a tree diagram, one or more times, only four graphs: a self energy graph and three tadpoles (fig. 1). In other words the four graphs in fig. 1

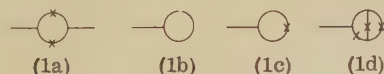


Fig. 1. The four primitive graphs of a dual theory.

represent all the proper n -point functions in a dual theory, all other graphs being one particle reducible.

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Any duality diagram [2] can be put in a one-to-one correspondence with either a sphere or a connected sum of tori, or a connected sum of projective planes with windows and points on it. Some of the points demarcate sides of the windows, while the others correspond to closed loops of related Feynman-like diagrams. The sides of the windows correspond to the external legs of the duality diagram. A given triangulation of the surface is related to a well-defined Feynman-like diagram [2][†]. In the case of a planar diagram (sphere with one window and points), it is known that there exists a triangulation which corresponds to the iteration of a planar tadpole. We show that in the other cases a special triangulation can be found with similar properties.

We start from a surface with windows and N points. Let us first reduce the number of points: a point can be brought near a given side of a window, and we can absorb it together with that side by the triangulation of fig. 2a. This triangulation

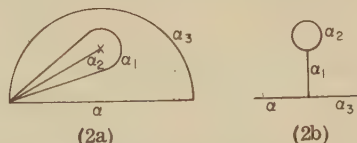


Fig. 2a) Triangulation of a point and the side of a window. b) The corresponding Feynman-like diagram.

lation means that a planar tadpole has been inserted on the external leg corresponding to the side of the window that we have used (fig. 2b). We are now brought back to the case of the same surface, with the same type of window, but with $N-1$ points. By repeating this process N times, we can get rid of all the points, and insert N planar tadpoles on external lines. Notice that

[†] See footnote [†] on next page.

these N tadpoles can be distributed in an arbitrary fashion on the external legs.

Let us now reduce the number of sides of a given window. This is done by the triangulation of fig. 3a, which means that we have arranged

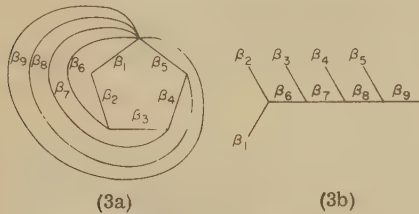


Fig. 3 a) Triangulation of a window.
b) The corresponding Feynman-like diagram.

the external lines corresponding to the sides of the window in a multiperipheral tree-configuration (fig. 3b). We carry out this procedure for all the windows of the surface, and thus are left with the same surface, with the same number of windows, each of them having now only one side (β_9 in fig. 3a).

We shall now reduce the number of one-sided windows. We bring two of them together and perform the triangulation of fig. 4a. Lines γ_1 and

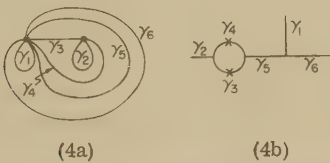


fig. 4 a) Triangulation of two one-sided windows.
b) The corresponding Feynman-like diagram.

γ_2 correspond either to external lines or to lines analogous to α_3 (fig. 1b) or β_9 (fig. 3b). The two lines γ_3 and γ_4 (fig. 4b) are twisted†. We encounter here the second type of irreducible graph, a self-energy diagram with twisted lines. After this triangulation, the two windows have been absorbed into a new larger one-sided window.

We repeat this process by absorbing this new window with a third one. If the windows are on a sphere, they can all be observed, and the triangulation is completed. It gives a chain of self-energy graphs inserted between multi-peripheral trees or on external legs, plus planar tadpoles which are hooked in arbitrary places on the trees.

† We recall that to each triangle of fig. 2a, 3a, 4a, 5a, 6a corresponds a vertex in fig. 2b, 3b, 4b, 5b, 6b, respectively. For the relation between a twist and the orientation of a triangle, see ref. 2.

If the surface is not a sphere, this process of triangulation stops when we are left with only one window. So, we now consider the two other possible types of surfaces, i.e., a connected sum of tori with one one-sided window or a connected sum of projective planes with one one-sided window. In the first case we use the canonical polygon representation of the connected sum of n tori with one window w [3].

$$\phi_1 \psi_1 \phi_1^{-1} \psi_1^{-1} w \phi_2 \psi_2 \phi_2^{-1} \psi_2^{-1} \dots \phi_n \psi_n \phi_n^{-1} \psi_n^{-1}$$

A triangulation of the surface for the particular case $n = 3$ is shown in fig. 5a. The Feynman-like

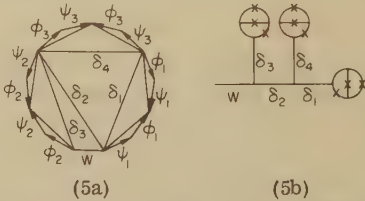


Fig. 5 a) Triangulation of a sum of tori with one-sided window.
b) The corresponding Feynman-like diagram.

diagram corresponding to this triangulation is shown in fig. 5b. A similar triangulation exists for all n . This canonical triangulation gives a Feynman-like diagram which is an iteration of our third one particle irreducible graph, a non-planar orientable tadpole with two loops.

The remaining type of surface is a sum of m projective planes with one one-sided window w' . The canonical polygon representation of this surface is

$$\chi_1 \chi_1 w' \chi_2 \chi_2 \chi_3 \chi_3 \dots \chi_m \chi_m$$

The case $m = 4$ is drawn in fig. 6a, together with

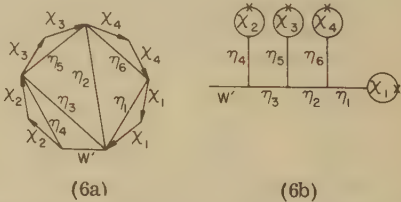


Fig. 6 a) Triangulation of a sum of projective planes with one one-sided window.
b) The corresponding Feynman-like diagram.

a triangulation of the surface. This triangulation is easily generalized to all m . It gives a Feynman-like diagram (fig. 6b), which is an iteration of the fourth and last one particle irreducible

graph, the non-planar non-orientable one-loop tadpole.

We have now proved the following theorem: any dual diagram can be reduced to the insertion in a tree of the four primitive one-particle irreducible graphs of fig. 1. The tadpoles can be moved to any place on the tree, and there may be any number of them. Non-planar self-energy graphs are locked between trees or on external lines, and there are at most $L-1$ of them if there are L external legs. The two tadpoles in fig. 1c and 1d never occur together in this reduction: if the diagram is non-orientable, only 1c occurs, if it is orientable, only 1d occurs. The identity of fig. 7b shows that a Feynman-like diagram

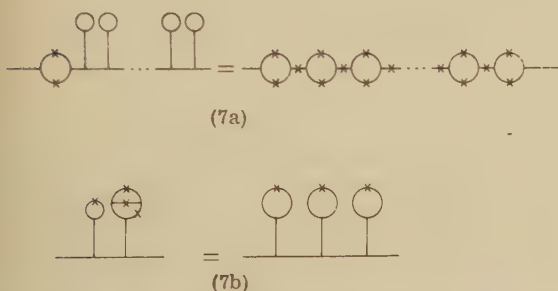


Fig. 7. Two identities between Feynman-like diagrams.

containing tadpoles 1c and 1d can in fact be drawn by using only the tadpole 1c. There is a one-to-one correspondence between these reduced graphs and the set of duality diagrams as defined in [2], so that no two of these reduced graphs are reducible to each other by a duality transformation.

This reduction of Feynman-like diagrams to a canonical form is valid for any dual model[†] and has two important applications.

1) The renormalizability of a dual-resonance model reduces to the study of the renormalizability

of the four irreducible operators corresponding to the graphs in fig. 1a-d, and of their iteration. Notice that the non-planar self-energy graph in fig. 1a need not be iterated if the iteration runs over planar and non-planar tadpoles. On the other hand, thanks to the identity of fig. 7a, the iteration of planar tadpoles can be turned into the iteration of self-energy graphs, if there is at least one of them. It is remarkable that the renormalization of dual-resonance models can be expressed only as a mass renormalization.

In the particular case of the generalized Veneziano model, we have already found [4,5] that the planar tadpole of fig. 1b and the non-orientable tadpole of fig. 1c can be renormalized by using elliptic functions. However, we have found that graph 1a exhibits peculiar branch points due to the fact that its divergence depends on its momentum. It is worth noting that, if we include internal quantum numbers or draw quark lines, this graph occurs only in channels with vacuum quantum numbers. If the Pomeron-chuk is to appear in the model, it is clear that new singularities would be expected to occur in this graph, but it is not clear that the ones we have found are physically acceptable.

2) Another application of the reduction of dual diagrams is the possibility of a formal summation of the whole perturbation series. In ref. 5, we show that the set of one-loop diagrams satisfies perturbative unitarity in second order if each irreducible diagram is counted once. If the same counting problem can be solved for more complicated diagrams, the perturbation series could be formally summed.

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[†] It is evident that, due to its topological nature, this theorem will remain valid in the presence of internal quantum numbers. The irreducible operators that appear in fig. 1 will however carry additional labels referring to the quantum numbers of the one particle states of the model.

DIVERGENCES AND RENORMALIZATION IN MULTILoop DUAL DIAGRAMS

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We analyse the new divergences which have recently been discovered in multiloop planar dual diagrams. We show that they can be removed by defining a counterterm for each order in the coupling constant g^2 . The physical interpretation of the counterterms as a mass renormalization is maintained at all orders in g^2 .

The exponential divergence [1] appearing in the one-loop planar dual diagrams has been removed by a procedure compatible with duality, unitarity, Regge behavior [2] and factorization [3]. This procedure has been interpreted as a mass renormalization, and it gives a physically sensible meaning to the unitarization program [4] of the dual-resonance model at the one-loop level. It is thus important to investigate the divergence and renormalization properties of the multiloop planar diagrams.

Since all the planar diagrams can be put into the form of an iteration of planar tadpoles, what is really needed is the planar tadpole operator [5] T_1 (fig. 1):



Fig. 1. Tadpole graph.

$$T_1 = 4\pi^2 g^2 \int_0^1 \frac{dw}{\ln^2 w} w^{-\alpha_0-1} f^{-4}(w) \langle 0 | \exp[(a | (1-w) E(1-w) | a)]$$

and its counterterm $\tilde{T}_1$, which is obtained from T_1 by replacing E by $\tilde{E}$, and which differs from T_1 by terms of order q^2 , where $q = \exp(2\pi^2/\ln w)$:

$$\tilde{T}_1(w) \xrightarrow{q \rightarrow 0} T_1(w) [1 + O(q^2)].$$

(The notations are those of ref. [5].)

The divergence of the dumbbell graph (fig. 2) have been investigated by Kaku and Scherk [6]. They

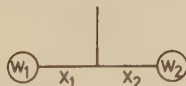


Fig. 2. Dumbbell graph.

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find that, besides the partition functions $f(w_1)$ and $f(w_2)$ associated with each planar tadpole, an infinite number of new partition functions appear when summing over the intermediate states. However, only one of them ($f(w_3)$, say), leads to a new divergence. The variable w_3 is a function of the variables w_1 and w_2 of each tadpole and of the variables x_1 and x_2 of the propagators, and w_3 can reach 1 for w_1 and w_2 different from 1. These authors, together with Yu, have analysed more complicated objects, and find 7 divergent partition functions in the 3-loop case, and 15 in the 4-loop case. It is not hard to see that the simple replacement of T_1 by $T_1 - \tilde{T}_1$ does not cancel the new divergences.

But this does not mean that the situation is hopeless: for instance, in the 4-point, 2-loop amplitude (fig. 3), there are triple poles in the variables s and t , and the residues of these triple poles clearly do

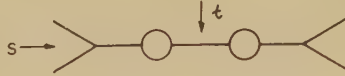


Fig. 3. 4-point two-loop graph.

not exhibit the new divergence, which comes only from the infinite summation over the states propagating between the two self-energy bubbles. Only the residues of the double poles in s and t have the new divergence, and so it may be possible to define a new mass-counterterm operator, of order g^4 , which would cancel in a physically significant way this new divergence. That this is indeed possible is what we are going to show. We also show that the same renormalization program can be extended to N -loop planar diagrams.

Let us first analyse from where the new divergence comes: from the topological identity of fig. 4, it

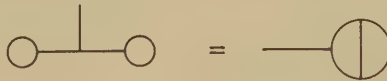


Fig. 4. Various identities between two-loop tadpoles.

is immediately clear that the two-loop planar tadpole has at least 3 divergences, since that number should be independent of the way this diagram is built from simpler diagrams: the divergence for $w_3 \rightarrow 1$ comes when one first does the trace on the line which runs around the rest of the diagram of the right hand side of the identity of fig. 4.

In general, in the N -loop tadpole, the same analysis reveals that there are at least $2^N - 1$ divergent partition functions[†]. However, there is only one essentially new divergence at each order; it is the one that comes from the path circling all the holes in the picture of Kikkawa and Sato [7], and this new divergence manifests itself only in the residues of the double poles in s and t for the 4-point, N -loop planar diagram. The basic idea is thus to first define a new tadpole counterterm of order $g^{2^N - 1}$ to cancel this essentially new divergence, and to cancel afterwards the other divergences by the lower-order tadpole counterterms. Let us first do this in the case of the dumbbell graph.

We explicitly exhibit the new divergence for $w_3 \rightarrow 1$ by one of the identities of fig. 5, in which the non-



Fig. 5. Various identities between two-loop tadpoles.

[†] The fact that there are no more than $2^N - 1$ divergences has been shown by the Berkeley group in the cases $N = 2, 3$, and 4. We assume this result to be also true for $N > 4$.

planar self-energy operator Σ_T appears $\dagger\dagger$. We now make use of the explicit form of Σ_T at zero momentum, computed in ref. [5]:

$$\Sigma_T(p=0) = \int_0^1 \frac{dx dy}{\ln^2 w_3} w_3^{-\alpha_0-1} f^{-4}(w_3) \langle 0 | \times \\ \exp[(a_1 | (1-w_3) E(1-w_3) | a_1) + (a_2 | (1-w_3) E(1-w_3) | a_2) + (a_1 | (1-w_3) E_T(1-w_3) | a_2)].$$

We use the crucial fact that

$$E_T \xrightarrow{q_3 \rightarrow 0} O(q_3^2) \quad \text{with } q_3 = \exp(2\pi^2/\ln w_3),$$

so that in this limit the oscillators a_1 and a_2 of the 2 lines entering the self-energy graph are completely decoupled. The second crucial point is that the matrix E is the same as the one of the planar tadpole operator. These two facts enable us to define a tadpole counterterm $\tilde{T}_2(w_1, w_2, w_3)^{\dagger\dagger\dagger}$ which cancels the divergence at $w_3 \rightarrow 1$ of the dumbbell $T_2(w_1, w_2, w_3)$. $\tilde{T}_2(w_1, w_2, w_3)$ is defined from the right-hand side of fig. 5 by replacing Σ_T by

$$\tilde{\Sigma}_T = 4\pi^2 g^2 \int_0^1 \frac{dx dy}{\ln^2 w_3} w_3^{-\alpha_0-1} f^{-4}(w_3) \langle 0 | \exp[(a_1 | (1-w_3) \tilde{E}(1-w_3) | a_1) + (a_2 | (1-w_3) E(1-w_3) | a_2)],$$

where a_1 are the oscillators of the free external line. This ensures first that

$$\tilde{T}_2(w_1, w_2, w_3) \xrightarrow{q_3 \rightarrow 0} T_2(w_1, w_2, w_3) [1 + O(q_3^2)]$$

so that $T_2(w_1, w_2, w_3) - \tilde{T}_2(w_1, w_2, w_3)$ is finite when $w_3 \rightarrow 1$.

Second, since the dependence of $\tilde{T}_2(w_1, w_2, w_3)$ on the external oscillators is exactly the same as the one of the tadpole counterterm $\tilde{T}_1$, it has the same duality properties as $\tilde{T}_1$, and gives rise to dual double poles only, when attached to an external tree: this allows us to consider $\tilde{T}_2(w_1, w_2, w_3)$ as inducing a mass renormalization of order g^4 .

We now come to the cancellation of the divergences at $w_2 \rightarrow 1$ and $w_1 \rightarrow 1$. As far as the dumbbell itself $T_2(w_1, w_2, w_3)$ is concerned, there is no problem: one just replaces the tadpole operators $T_1(w_1)$ and $T_1(w_2)$ by $\tilde{T}_1(w_1)$ and $\tilde{T}_1(w_2)$ in the left-hand side of fig. 5. One thus gets 3 new terms.

$$\tilde{T}_2(w_1, w_2, w_3) = T_2(w_1, w_2, w_3) [1 + O(q_1^2)]$$

$$\tilde{T}_2(w_1, w_2, w_3) = T_2(w_1, w_2, w_3) [1 + O(q_2^2)]$$

$$\tilde{T}_2(w_1, w_2, w_3) = T_2(w_1, w_2, w_3) [1 + O(q_1^2)] [1 + O(q_2^2)]$$

so that $T_2(w_1, w_2, w_3) - \tilde{T}_2(w_1, w_2, w_3) - \tilde{T}_2(w_1, w_2, w_3) + \tilde{T}_2(w_1, w_2, w_3)$ is finite when $w_1 \rightarrow 1$ and/or $w_2 \rightarrow 1$.

Obviously, one has to do something similar on $\tilde{T}_2(w_1, w_2, w_3)$. Unfortunately, in the definition of $\tilde{T}_2(w_1, w_2, w_3)$, the symmetry between w_1 and w_2 is no longer apparent. But this symmetry can be restored, because we did not replace E by $\tilde{E}$ in the quadratic form in a_2 of $\tilde{\Sigma}_T$: the trace on the oscillators a_2 is thus formally exactly the same as the trace occurring in the dumbbell without external lines, and one can use the identities of fig. 6 to come back to a form symmetric in w_1 and w_2 . This symmetric

$\dagger\dagger$ As discussed in ref. [5], Σ_T has periodicities in the variable x which restrict the integration region. This restriction is not obvious on the dumbbell form of T_2 and was not exhibit in ref. [6], but it does not alter the conclusions of that paper. It appears only in the terms depending on the external oscillators. We hope to discuss this and the technical details in a subsequent paper.

$\dagger\dagger$ By underlining w_3 , we mean that we are taking a suitable asymptotic form of T_2 when $w_3 \rightarrow 1$. For the subsequent cancellation of the divergences at $w_1 \rightarrow 1$ and for $w_2 \rightarrow 1$, it is important to remember which divergences an individual counterterm cancels.



Fig. 6. Vacuum fluctuation identities.

form then enables us to eliminate the divergences of $\tilde{T}_2(w_1, w_2, w_3)$ at $w_1 = 1$ and/or $w_2 = 1$ by defining:

$$\tilde{T}_2(w_1, w_2, w_3) = \tilde{T}_2(w_1, w_2, w_3) [1 + O(q_1^2)]$$

$$\tilde{T}_2(w_1, w_2, w_3) = \tilde{T}_2(w_1, w_2, w_3) [1 + O(q_2^2)]$$

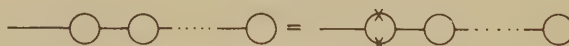
$$\tilde{T}_2(w_1, w_2, w_3) = \tilde{T}_2(w_1, w_2, w_3) [1 + O(q_2^2)] [1 + O(q_1^2)],$$

exactly as one does for the complete dumbbell $T_2(w_1, w_2, w_3)$ itself. These new terms have the same dependence on the external oscillators as $\tilde{T}_2(w_1, w_2, w_3)$. This similarity in the operations on $T_2(w_1, w_2, w_3)$ and $\tilde{T}_2(w_1, w_2, w_3)$, together with the identities of fig. 5 ensures that the sum of the 8 terms:

$$T_2(w_1, w_2, w_3) - \tilde{T}_2(w_1, w_2, w_3) - \tilde{T}_2(w_1, w_2, w_3) + \tilde{T}_2(w_1, w_2, w_3) \\ - \tilde{T}_2(w_1, w_2, w_3) + \tilde{T}_2(w_1, w_2, w_3) + \tilde{T}_2(w_1, w_2, w_3) - \tilde{T}_2(w_1, w_2, w_3)$$

is finite in all the limits of one or more w 's going to 1. The new tadpole counterterms of order g^3 , is in fact the sum of 4 terms; it is divergent only for $w_3 \rightarrow 1$, and has the same duality properties as the tadpole counterterm in order g , and thus the same physical interpretation of a mass renormalization, but in order g^4 .

This method can be extended to N loops; one first exhibits the essentially new divergence by the identity of fig. 7, cancel it by the same procedure as the w_3 divergence in the 2-loop case, and after-

Fig. 7. Identity between N -loop tadpoles.

wards, cancel the other divergences by the counterterms defined for an order $\leq N - 1$. Writing explicitly the counterterm of order g^{2N-1} becomes quickly tedious, since the number of its terms increases quickly; but the various topological identities generalizing those of fig. 5 and fig. 6 will guarantee the overall cancellation of all the divergences*.

We can thus show that the set of planar dual diagrams is a renormalizable set**. All the counterterms have the same structure that enables us to interpret them as a mass renormalization.

The same new divergences occur in the iteration of the non orientable tadpole, but the available topological identities did not enable us to exhibit and renormalize them in the same way as in the planar case. But since the structure of the planar and non orientable tadpoles are very similar, it is likely that more powerful methods (and for example the use of functionals on Riemann surfaces [8]) could solve these problems in a global way.

We are grateful to Prof. C. Bouchiat for many discussions and one of us (A.N.) would like to thank the Laboratoire de Physique Théorique et Hautes Energies d'Orsay for its hospitality during the summer of 1970, where this work was done.

* One also has to use the fact that all the 2^{N-1} divergences cannot occur simultaneously: two of them can appear simultaneously only if there exists one topological configuration of the dual diagram which exhibits these two divergences (J. Scherk: private communication). We thank Prof. J.H. Schwarz for a useful discussion on this subtle point.

** Since the iteration of a finite counterterm (see ref. 2) introduces the same new divergences as the iteration of the infinite counterterms, it makes its interpretation as a finite change in the Born term rather awkward.

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TACHYON-FREE DUAL MODEL WITH A POSITIVE-INTERCEPT TRAJECTORY*

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A dual-resonance model with two kinds of trajectories split by one-half unit is constructed. The lowest mass state on the leading trajectory has spin one, although the G -parity assignments are unsuitable for an identification with π 's and ρ 's.

One of the important unsolved problems in the construction of a realistic dual-resonance model for mesons is the introduction of tachyon-free positive-intercept trajectories. In this letter we report some progress in this direction by exhibiting a model with two kinds of trajectories, $\alpha_1(s)$ and $\alpha_2(s)$, such that

$$\alpha_1(s) = \alpha_2(s) + 1/2 = s + 1 - m^2 \quad (1)$$

and the first state on $\alpha_1(s)$ is a vector meson at $s = m^2$. These two features are just what one would like for a dual model of the π - ρ system. Unfortunately, the model assigns negative G -parity to α_1 and positive G -parity to α_2 so that an identification with π 's and ρ 's cannot be made. Conceivably a similar scheme might have the physical G -parity, although the lack of an underlying quark description of the states makes it doubtful that the ultimate model will be constructed along the same lines.

In addition to the harmonic oscillator annihilation operators a_n^μ of the standard Veneziano model [1], we introduce anticommuting operators b_m^μ , where the index m runs over half integers from $1/2$ to $+\infty$. The following algebra is postulated (our metric is spacelike):

$$[b_m^\mu, a_n^\nu] = [b_m^{\mu\dagger}, a_n^\nu] = \{b_m^\mu, b_n^\nu\} = 0 \quad (2a)$$

$$[a_m^\mu, a_n^{\nu\dagger}] = \{b_m^\mu, b_n^{\nu\dagger}\} = g^{\mu\nu} \delta_{mn} \quad (2b)$$

Next we construct the operators

$$H^\mu(\tau) = \sum_{m=-\infty}^{\infty} b_m^\mu \exp(im\tau) \quad (3)$$

$$L_n^b = -\frac{i}{4\pi} : \int_{-\pi}^{\pi} d\tau \exp(in\tau) H(\tau) \cdot \dot{H}(\tau) : \quad (4)$$

where the sum in (3) runs over half integers, $b_{-m}^\mu \equiv b_m^{\mu\dagger}$, and $\dot{H}^\mu(\tau) = dH^\mu(\tau)/d\tau$. The following algebra is readily verified:

$$[L_m^b, L_n^b] = (n-m) L_{m+n}^b \quad (5)$$

$$\{H^\mu(\tau), H^\nu(\tau')\} = 2\pi g^{\mu\nu} \delta(\tau - \tau') \quad (6)$$

$$[L_n^b, H^\mu(\tau)] = \exp(in\tau) \left(\frac{n}{2} H^\mu(\tau) - i \dot{H}^\mu(\tau) \right). \quad (7)$$

From these formulas one observes that L_0^b and $L_{\pm 1}^b$ are the infinitesimal generators [2] of fractional-linear transformations in the variable $z = \exp(i\tau)$, as required for the construction of Möbius-invariant N -point functions.

In order to construct N -point functions in a way that makes factorization and cyclic symmetry manifest simultaneously, we introduce a vertex operator in the Koba-Nielsen variables [3]. For the emission of a vector meson of momentum k_μ and polarization vector ϵ_μ satisfying $k^2 = -m^2$ and $k \cdot \epsilon = 0$ we use

$$V(k, z) = \epsilon \cdot H(z) V_0(k, z) \quad (8)$$

where V_0 is the vertex operator of the standard Veneziano model

$$V_0(k, z) = z^{-2k \cdot p} \exp(ik \cdot x) \exp \left\{ \sqrt{2} \sum_{n=1}^{\infty} k \cdot a_n^\dagger \frac{z^n}{\sqrt{n}} \right\} \exp \left\{ \sqrt{2} \sum_{n=1}^{\infty} k \cdot a_n \frac{z^{-n}}{\sqrt{n}} \right\}. \quad (9)$$

p and x are momentum and position operators sometimes represented by "zero modes" and a delicate limiting procedure [2]. The N -point function for vector mesons is given by

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$$A_N = \int d\mu_N(z) \prod_{i=1}^N \left\{ z_i^{-1/2} |z_i - z_{i+1}|^{1/2 - m^2} \right\} \\ \times \langle 0 | \prod_{i=1}^N V(k_i, z_i) | 0 \rangle, \quad (10)$$

where $z_{N+1} \equiv z_1$ and $d\mu_N(z)$ is the usual Möbius-invariant volume element

$$d\mu_N(z) = (z_2 - z_N) dz_3 dz_4 \dots dz_{N-1} \prod_{i=2}^{N-1} \left(\frac{\theta(z_i - z_{i+1})}{z_i - z_{i+1}} \right). \quad (11)$$

with z_1, z_2, z_N arbitrarily chosen as fixed points. Cyclic symmetry follows from the Möbius-invariance of the integrand and the commutation properties of V with itself*. Factorization is best demonstrated by recasting eq. (10) into the form

$$A_N = \langle 0 | \epsilon_1 \cdot b_{1/2} V(k_2, 1) D V(k_3, 1) \dots \\ D V(k_{N-1}, 1) \epsilon_N \cdot b_{1/2}^\dagger | 0 \rangle \quad (12)$$

where the propagator D is given by

$$D = \int_0^1 dx (1-x)^{-m^2-1/2} x^{m^2+L_0-3/2} \quad (13)$$

and

$$L_0 = p^2 + \sum_{n=1}^{\infty} n a_n \cdot a_n + \sum_{m=1/2}^{\infty} m b_m^\dagger \cdot b_m. \quad (14)$$

It is easy to see that A_N vanishes unless N is even because the calculation involves pairing the b operators. In particular, one finds for the four-point function

$$A_4 = \int_0^1 dx x^{m^2-s-1} (1-x)^{m^2-t-1} \left[(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4) \sqrt{\frac{1-x}{x}} + \right. \\ \left. (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot \epsilon_4) \sqrt{\frac{x}{1-x}} - (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4) \sqrt{x(1-x)} \right]. \quad (15)$$

The first pole of this amplitude is a scalar at $s = m^2 - 1/2$ corresponding, in the notation of eq. (1), to $\alpha_2(s) = 0$. The general structure of the spectrum can be deduced from the eigenstates and eigenvalues of the Hamiltonian $m^2 - 1/2 + L_0$.

* In the anticommutation of H 's the δ functions that arise are evaluated at the edges of the integration region and make no contribution.

One notices that states with an even number of b excitations lie on $\alpha_2(s)$ or one of its daughters whereas states with an odd number of b excitations lie on $\alpha_1(s)$ or one of its daughters. Hence one introduces the G -parity operator

$$G = (-1)^{\sum_{m=1/2}^{\infty} b_m^\dagger \cdot b_m} \quad (16)$$

which anticommutes with the vertex V . Thus the α_1 states have odd G -parity and the α_2 states have even G -parity. This is opposite from what is required for the ρ - π system. The degeneracy of the spectrum, which is qualitatively rather similar to the usual case, can also be deduced from the Hamiltonian. One finds, in particular, that the leading Regge trajectory $J = \alpha_1(s)$ is non-degenerate. The next one, $J = \alpha_2(s)$, has a single scalar particle at $\alpha_2(s) = 0$, but as many as four states along the rest of the trajectory, three of them having the natural parity.

In the case $m^2 = -1/2$ the operators L_n^b can be added to the L_n 's of Virasoro [4] and used to generate an infinite set of Ward identities. This mass value corresponds to $\alpha_1(0) = 3/2$ and $\alpha_2(0) = 1$, which is not surprising once one notices that the amplitude for scattering N of the lowest-mass even G -parity scalars is just the usual Veneziano formula. (The odd- G states do modify the corresponding loop formulas, however.)

In conclusion, our model has all the difficulties of the simpler one. It does, however, succeed in incorporating additional "odd G " trajectories whose leading state is a vector meson without introducing too many new problems or drastically increasing the degeneracy of the spectrum. As our understanding of the algebraic requirements of duality improves one may hope that more realistic models for meson scattering can eventually be formulated.

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FACTORIZABLE DUAL MODEL OF PIONS*

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Abstract: A new dual-resonance model of mesons is constructed through the use of creation and annihilation operators having simple anticommutation properties as well as harmonic-oscillator type operators of the conventional model. This model has the following virtues not shared by the conventional one: (i) The leading trajectory (ρ - f^0) does not make a particle when it passes through zero. (ii) A π -trajectory lies one-half unit below the ρ -trajectory. (iii) Trajectories with abnormal-parity couplings to pions, such as ω - A_2 and η' , also occur. (iv) The π , ρ , f^0 , ω , A_2 and η' are all forced to have the proper G -parity and isospin. (v) The amplitudes for $\pi\pi \rightarrow \pi\pi$ and $\pi\pi \rightarrow \pi\omega$ are precisely the ones that have been suggested previously. (vi) The model contains neither embarrassing unobserved low-mass states nor an excessive number of high-mass states. (vii) All the physical states that we have checked have positive norms.

To be fair, we should also list some shortcomings of our model: (a) Trajectories with normal-parity couplings occur one-half unit too high. Thus, for example, $\alpha_\rho(0) = 1$ and $\alpha_\pi(0) = \frac{1}{2}$, so that the π is a tachyon and the ρ is massless. On the other hand, the abnormal-parity trajectories are nicely located. For example, $\alpha_\omega(0) = \frac{1}{2}$ and $\alpha_{\eta'}(0) = -1$. (b) A straightforward generalization to $SU(3)$ is unsatisfactory because of the non-degeneracy of the ρ and ω . These features of the model clearly indicate that it is not accurately describing the real world. Perhaps its most important property is that it contains a gauge algebra *larger* than the Virasoro algebra of the conventional model. We believe that an understanding of this algebra may prove to be very important in the construction of more realistic models.

1. INTRODUCTION

In the nearly three years since Veneziano's classic paper [1] a great deal of effort has been expended on attempts to construct dual-resonance models that either bear a resemblance to reality or else could serve as the basis for a full-fledged unitary (albeit unrealistic) theory of hadrons. Considerable progress toward the second goal has grown out of the study of

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the N -point generalizations [2] of Veneziano's beta function. First, the one-loop graphs were constructed [3] and renormalized [4]. More recently, the general N -loop graphs have been constructed [5] and (in the planar case) renormalized [6]. Scherk has furthermore shown [7] that the renormalization is essentially unique when the intercept of the leading Regge trajectory $\alpha_0 = 1$, whereas in all other cases there is infinite ambiguity, i.e., the theory is non-renormalizable. The special role of $\alpha_0 = 1$ was pointed out earlier by Virasoro [8] in connection with the problem of negative-norm states (ghosts). He showed that in this case there is a large gauge algebra that probably allows for the cancellation of all ghosts.

We shall refer to the generalized Veneziano model with $\alpha_0 = 1$ as the "conventional model". Despite the manifold virtues of this model, the unitarization program encounters at least two serious problems. First, the model contains a tachyon, which causes severe difficulties, of course. Second, certain non-planar loop graphs give rise to a new singularity [9] that is not simply related to single-particle states. It has been suggested (probably correctly) that this singularity should be identified as the pomeron. Its precise character and intercept are determined by detailed characteristics of the spectrum of single-particle states. In the conventional model it turns out to be a branch point having intercept $\frac{1}{4}$ (when the linear dependences are properly accounted for). It is our strong impression that such a singularity can be reconciled with unitarity only when it is a pole. One possible attitude, closely akin to the bootstrap philosophy [10], is that of all the many dual models that might be constructed, only the "right" one is free of ghosts and tachyons and gives a pomeron *pole*. Veneziano has further suggested [11] that even the coupling constant may be determinable. However, this is an attitude we do not share.

Another shortcoming of the conventional model is its lack of resemblance to the real world. At first, attempts to incorporate more realistic quantum numbers centered on attaching superstructure describing quark spin and internal quantum numbers to the conventional model [12]. This approach encountered insuperable difficulties relating to parity doubling and ghosts. Recently, a number of more ambitious proposals have been made, but none of them has been very successful so far.

This paper presents a new model containing realistic quantum numbers without parity doubling or ghosts (as far as we can tell). The basic algebra of the model has been previously reported [13]. However, the version presented in ref. [13] contains "pions" and "rhos" of the wrong G -parity, and also has a number of other inadequacies. Shortly after submitting that work we discovered a slight but crucial modification that allows for substantial improvement.

The new model has pions and rhos with the correct quantum numbers. It also contains analogs of the Virasoro gauges of the conventional model, and it appears to be free of ghosts. More remarkably, it possesses additional gauge operators which have the almost incredible property of decoupling the ground state and the leading Regge trajectory! The algebra that is operative is not yet fully understood, but it is undoubtedly rather deep. The model contains a π , ρ , f^0 , ω , A_2 and η' , all with the proper quantum num-

bers. It does *not* contain low-mass states without an experimental identification. In fact, it does not even have a σ degenerate with the ρ or an A_1 . For higher masses the degeneracy probably grows exponentially with mass as in the conventional model, although the gauge algebra is so powerful that it is not yet clear whether it grows that fast. The $\rho - f^0$ and $\omega - A_2$ trajectories are not "doubled", contrary to previous suggestions [14]. The amplitude for $\pi\pi \rightarrow \pi\pi$ is the one suggested by Shapiro and Lovelace [15], while the one for $\pi\pi \rightarrow \pi\omega$ precisely coincides with Veneziano's original suggestion [1].

An empirical rule observed in the model is that those trajectories with normal-parity couplings to pions (π , ρ , f^0 , etc.) occur one-half unit too high, whereas those with abnormal-parity couplings to pions (ω , A_2 , η' , etc.) are nicely located. This means that we have $\alpha_\pi(0) = \frac{1}{2}$, $\alpha_\rho(0) = \alpha_{f^0}(0) = 1$, $\alpha_\omega(0) = \frac{1}{2}$, and $\alpha_{\eta'}(0) = -1$. This placement of trajectories implies that the π is a tachyon and the ρ is massless. Another difficulty is that the non-degeneracy of the ρ and ω makes a straightforward generalization to $SU(3)$ most unsatisfying. Namely, it would result in two sets of $\rho - f$ and $\omega - A_2$ trajectories split by one-half unit. One set would carry the normal couplings and the other the abnormal ones. In our opinion, the model should be regarded as intrinsically limited to $SU(2)$. We suspect that a future better model might require including $SU(3)$ as a broken symmetry.

The paper is organized as follows: sect. 2 is a review of the conventional model stressing those approaches and notations that are required in subsequent sections. Sect. 3 presents the proposed new model. In particular, it explains why the masses are constrained to be those mentioned above. Sect. 4 discusses the basic conjecture concerning the existence of extra gauge operators. The conjecture is not proved, but a strong case is made for it. In sect. 5 there is a rather detailed investigation of the spectrum. All the states at the first few mass levels and on the leading trajectories are discussed. In sect. 6 the formulas for the six-pion amplitude and the $\pi\rho \rightarrow \pi\rho$ amplitudes are presented. Sect. 7 is the conclusion.

2. REVIEW OF CONVENTIONAL DUAL-RESONANCE MODEL

The spectrum of the conventional dual-resonance model is conveniently described in the Fock space generated by four-vector creation operators $a_m^\dagger{}^\mu$, $m = 1, 2, \dots$ (ref. [16]). These operators satisfy the commutation relations

$$[a_m^\mu, a_n^\nu] = [a_m^\dagger{}^\mu, a_n^\dagger{}^\nu] = 0, \quad (2.1a)$$

$$[a_m^\mu, a_n^\dagger{}^\nu] = -g^{\mu\nu} \delta_{m,n}, \quad (2.1b)$$

where we use the timelike metric $g^{00} = -g^{ii} = 1$. We find it helpful to replace the a -operators and the momentum operator p by

$$\alpha_n^\mu = -i\sqrt{n} a_n^\mu, \quad n = 1, 2, \dots, \quad (2.2a)$$

$$\alpha_{-n}^\mu = i\sqrt{n} a_n^{\dagger\mu}, \quad n = 1, 2, \dots, \quad (2.2b)$$

$$\alpha_0^\mu = \sqrt{2} p^\mu. \quad (2.2c)$$

The algebra of eq. (2.1), re-expressed for the α -operators, takes the form

$$[\alpha_m^\mu, \alpha_n^\nu] = -m g^{\mu\nu} \delta_{m+n, 0}. \quad (2.3)$$

Following Ramond [17], we define the operator

$$P^\mu(\tau) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in\tau}, \quad (2.4)$$

whose τ space average is just the momentum:

$$p^\mu = \langle P^\mu(\tau) \rangle = \frac{1}{2\pi} \int_0^{2\pi} P^\mu(\tau) d\tau. \quad (2.5)$$

The Fourier moments of $P^2(\tau)$ are of particular importance:

$$L_n^a = -\langle e^{in\tau} : P^2(\tau) : \rangle = -\frac{1}{2} : \sum_{m=-\infty}^{\infty} \alpha_{-m} \cdot \alpha_{n+m} :. \quad (2.6)$$

We use the superscript "a" as a reminder that an operator is defined in terms of the a oscillators since in the next section we shall introduce a "b" space as well. For $n = 0$, eq. (2.6) becomes

$$L_0^a = -\frac{1}{2} \alpha_0 \cdot \alpha_0 - \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m = R_a - p^2, \quad (2.7)$$

while for $n > 0$,

$$L_n^a = - \sum_{m=0}^{\infty} \alpha_{-m} \cdot \alpha_{m+n} - \frac{1}{2} \sum_{m=1}^{n-1} \alpha_m \cdot \alpha_{n-m}. \quad (2.8)$$

The operators L_n^a have the elegant algebra

$$[L_m^a, L_n^a] = (m-n) L_{m+n}^a, \quad (2.9)$$

except when $m+n = 0$ an uninteresting constant may need to be added to the commutator. In particular, L_{-1}^a , L_0^a and L_1^a form an $SU(1, 1)$ algebra [18].

The construction of N -point functions can be achieved in a number of ways. The one that is most convenient for our purposes is to define a vertex operator

$$V_0(k, z) = e^{ik \cdot x} e^{\sqrt{2}k \cdot \sum_{n=1}^{\infty} \frac{\alpha_{-n} z^n}{n}} e^{-\sqrt{2}k \cdot \sum_{n=1}^{\infty} \frac{\alpha_n z^{-n}}{n}} z^{-2k \cdot p}, \quad (2.10)$$

to describe the emission of a particle of four-momentum k^μ satisfying a mass-shell condition $k^2 = m^2$. The N -point function is then given by [19]

$$A_N(k_1, k_2, \dots, k_N) = \int d\mu_N(z) \left\{ \prod_{i=1}^N |z_i - z_{i+1}|^{-m^2} \right. \\ \left. \times \langle 0 | V_0(k_1, z_1) V_0(k_2, z_2) \dots V_0(k_N, z_N) | 0 \rangle \right\}, \quad (2.11)$$

where $z_{N+1} \equiv z_1$ and

$$d\mu_N(z) = (z_2 - z_N) dz_3 dz_4 \dots dz_{N-1} \prod_{i=1}^{N-1} \frac{\theta(z_i - z_{i+1})}{z_i - z_{i+1}}. \quad (2.12)$$

The integrations in eq. (2.11) are along the real axis with limits controlled by the θ -functions of eq. (2.12); z_1, z_2 and z_N are fixed numbers satisfying $z_1 > z_2 > z_N$. The most important feature of A_N is its invariance under cyclic permutation of the momenta (duality). This fact depends critically on two properties of eq. (2.11). They are (1) the invariance of the integrand under Möbius transformations and (2) the rule for commuting $V_0(k_i, z_i)$ with $V_0(k_j, z_j)$.

Consider a Möbius transformation

$$z \rightarrow z' = Tz = \frac{az + b}{cz + d},$$

with a, b, c, d real numbers satisfying $ad - bc = 1$. These transformations form the group $O(2, 1)$, or, what is essentially the same thing, $SU(1, 1)$.

When all the variables $z_1, z_2, \dots, z_N$ appearing in eq. (2.11) are simultaneously subjected to the *same* transformation, the resulting expression has the same form with $z'_1, z'_2, \dots, z'_N$ replacing $z_1, z_2, \dots, z_N$. To prove this one first does an easy calculation to show that the measure is form invariant, i.e., $d\mu_N(z) = d\mu_N(z')$. The invariance of the rest of the integrand is demonstrated by constructing a unitary operator

$$\Lambda(T) = e^{i\xi_T L^a} \quad (2.13)$$

where L^a is made from $L_0^a, L_{\pm 1}^a$ and ξ_T is a three-vector appropriate for T . The vertex operator transforms by

$$\Lambda(T) V_0(k, z) \Lambda^{-1}(T) = (a - cz')^{-2m^2} V_0(k, z'). \quad (2.14)$$

The most convenient way to prove (2.14) is by first considering the infinitesimal transformations described by the formula [19, 20]

$$[L_n^a, V_0(k, z)] = z^n \left(z \frac{d}{dz} - nm^2 \right) V_0(k, z), \quad (2.15)$$

valid for all n . The proof of Möbius invariance is completed by noting that $e^{i\xi_T L^a}$ leaves the zero-momentum ground state, $|0\rangle$, invariant, and that

the factors $\prod_{i=1}^N (a - cz')^{-2m^2}$ arising from the transformation of the vertices

are compensated for by extra factors arising in the transformation of $\prod_{i=1}^N |z_i - z_{i+1}|^{-m^2}$. The Möbius invariance of the integrand implies, in particular, that A_N does not depend on the values chosen for z_1 , z_2 and z_N .

The proof of cyclic symmetry is completed by choosing the Möbius transformation T such that

$$z'_1 = z_N, \quad z'_2 = z_1, \quad z'_N = z_{N-1}.$$

Then, in order to obtain the formula for $A_N(k_N, k_1, k_2, \dots, k_{N-1})$, it is necessary to commute $V_0(k_N, z'_1)$ past $V_0(k_1, z_2) \dots V_0(k_{N-1}, z_N)$. The appropriate formula for this purpose is

$$V_0(k_i, z_i) V_0(k_j, z_j) = V_0(k_j, z_j) V_0(k_i, z_i) (-1)^{-2k_i \cdot k_j}. \quad (2.16)$$

Therefore, using momentum conservation and the mass shell condition, the complete commutation of $V_0(k_N, z'_1)$ gives rise to a phase $(-1)^{2m^2}$. Fortunately, this phase is cancelled by one occurring in the transformation of $\prod_{i=1}^N |z_i - z_{i+1}|^{-m^2}$ suitably redefined.

Another important property of A_N is its factorizability. This is demonstrated by using

$$V_0(k, z) = z^{L_0^a} z^{m^2} V_0(k) z^{-L_0^a}, \quad (2.17)$$

($V_0(k)$ is a shortened notation for $V_0(k, 1)$), and carefully letting $z_1 \rightarrow \infty$, $z_1 \rightarrow 1$, $z_N \rightarrow 0$. In this way, after a suitable change of variables, one obtains

$$A_N(k_1, k_2, \dots, k_N) = \langle 0; k_1 | V_0(k_2) D_0 V_0(k_3) \dots D_0 V_0(k_{N-1}) | 0; k_N \rangle, \quad (2.18)$$

where the propagator D_0 is given by

$$D_0 = \int_0^1 dx x^{L_0^a + m^2 - 1} (1-x)^{-m^2 - 1}, \quad (2.19)$$

and $|0; k\rangle$ represents a ground state with momentum k . (In the future we shall simply write $|0\rangle$, leaving the momentum implicit.)

A case of particular symmetry and beauty, discovered by Virasoro [8], occurs when $m^2 = -1$. In this case the leading Regge trajectory is $\alpha(s) = 1 + s$, and the ground state is a tachyon. Also, the propagator takes the simple form

$$D_0 = (L_0^a - 1)^{-1}, \quad (2.20)$$

so that the physical states satisfy the infinite-component wave equation

$$(L_0^a - 1)\varphi = 0. \quad (2.21)$$

Furthermore, the on-shell physical states are subject to the subsidiary conditions

$$L_n^a \varphi = 0, \quad n = 1, 2, \dots \quad (2.22)$$

To prove this it is sufficient to show that $L_0^a - L_n^a - 1$ annihilates a physical tree

$$D_0 V_0(k_2) D_0 V_0(k_3) \dots D_0 V_0(k_{N-1}) |0\rangle.$$

Using eq. (2.9) and eq. (2.15) one has

$$\begin{aligned} (L_0^a - L_n^a - 1) \frac{1}{L_0^a - 1} V_0(k) \\ = \frac{1}{L_0^a + n - 1} (L_0^a - L_n^a + n - 1) V_0(k) \\ = \frac{1}{L_0^a + n - 1} V_0(k) (L_0^a - L_n^a - 1). \end{aligned} \quad (2.23)$$

This calculation, together with the fact that $L_0^a - L_n^a - 1$ annihilates an on-shell ground state, proves the validity of the Virasoro subsidiary conditions. Notice that the gauge operators $L_0^a - L_n^a - 1$ annihilate physical states off mass shell as well as on mass shell, whereas L_n^a only annihilates on-mass-shell physical states.

One final construct of the conventional model that we shall need is the twist operator [21]. By definition, the twist operator, Ω_a , reverses the order of the external states along a tree, i.e., a tree and a twisted tree (see fig. 1) are related by

$$\begin{aligned} \Omega_a V_0(k_1) D_0 V_0(k_2) D_0 \dots D_0 V_0(k_{N-1}) |0\rangle \\ = V_0(k_N) D_0 V_0(k_{N-1}) D_0 \dots D_0 V_0(k_2) |0\rangle. \end{aligned} \quad (2.24)$$

A formula for Ω_a is

$$\Omega_a = (-1)^{R_a} e^{-L-1} = (-1)^{L_0 + p^2} e^{-L-1}. \quad (2.25)$$

A good way to prove that this formula satisfies eq. (2.24) is by letting it act on the integrand of a factorized form of eq. (2.11). Using the $SU(1,1)$

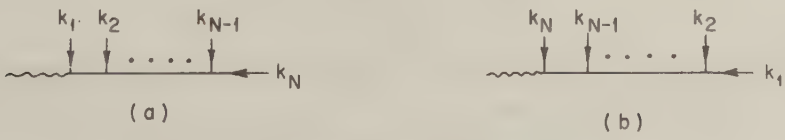


Fig. 1a, b. Two N -particle states related by twisting.

transformation rules, one shows in this way that

$$\begin{aligned}
 \Omega_a V_0(k_1, z_1) V_0(k_2, z_2) \dots V_0(k_N, z_N) |0\rangle \\
 = (-1)^{L_0^a + p^2} V_0(k_1, z_1^{-1}) V_0(k_2, z_2^{-1}) \dots V_0(k_N, z_N^{-1}) |0\rangle \\
 = (-1)^{p^2} V_0(k_1, 1-z_1) V_0(k_2, 1-z_2) \dots V_0(k_N, 1-z_N) |0\rangle \\
 = V_0(k_N, 1-z_N) V_0(k_{N-1}, 1-z_{N-1}) \dots V_0(k_1, 1-z_1) |0\rangle.
 \end{aligned} \quad (2.26)$$

The last step used eq. (2.16). The verification of twisting can now be completed by making the change of integration variables $z_i' = 1 - z_{N-i+1}$.

The most important fact about the twist operator for our purposes here lies in the observation that physical states are eigenstates of $\Omega^\dagger$,

$$\Omega^\dagger \varphi = (-1)^{R_a} e^{L_1^a} \varphi = (-1)^{R_a} \varphi. \quad (2.27)$$

Furthermore, when isospin is introduced by the Chan-Paton procedure [22] (which seems to be the only sensible one) and sums on all the permutations of external lines of tree graphs are included, the eigenvalue of $\Omega^\dagger$ may be identified with the charge conjugation quantum number of the neutral member of the multiplet under consideration. Therefore, if the G -parity of a physical state is known, its isospin may be readily deduced from its mass. The generalization of this procedure to $SU(3)$ is not difficult, but will not be required in this paper.

3. NEW DUAL-RESONANCE MODEL

The Fock space described in sect. 2 can be enlarged by the addition of anticommuting creation and annihilation operators. Specifically let us consider the space generated by the oscillators of the preceding section together with b_m^μ , $m = \frac{1}{2}, \frac{3}{2}, \dots$. We postulate, in addition to eq. (2.3)

$$\{b_m^\mu, b_n^\nu\} = -g^{\mu\nu} \delta_{m+n, 0}, \quad (3.1a)$$

$$[b_m^\mu, \alpha_n^\nu] = 0, \quad (3.1b)$$

where, as a matter of convenience, we have defined $b_{-m}^\mu = b_m^{\dagger\mu}$. We next introduce the operator

$$H^\mu(\tau) = \sum_{m=-\infty}^{\infty} b_m^\mu e^{-im\tau}, \quad (3.2)$$

with the sum running over all half integers, of course. This operator has the anticommutation relations

$$\{H^\mu(\tau), H^\nu(\tau')\} = -2\pi g^{\mu\nu} \delta(\tau - \tau'). \quad (3.3)$$

Letting $\dot{H}^\mu(\tau) = d/d\tau H^\mu(\tau)$, define

$$L_n^b = -\frac{1}{2}i \langle e^{in\tau} : H(\tau) \cdot \dot{H}(\tau) : \rangle = -\frac{1}{4} : \sum_{m=-\infty}^{\infty} (n+2m) b_{-m} \cdot b_{n+m} : . \quad (3.4)$$

For $n = 0$, eq. (3.4) becomes

$$L_0^b = - \sum_{m=\frac{1}{2}}^{\infty} m b_{-m} \cdot b_m = R_b , \quad (3.5)$$

while for $n = 1, 2, \dots$ we have

$$L_n^b = -\frac{1}{2} \sum_{m=\frac{1}{2}}^{\infty} (n+2m) b_{-m} \cdot b_{n+m} - \frac{1}{4} \sum_{m=\frac{1}{2}}^{n-\frac{1}{2}} (n-2m) b_m \cdot b_{n-m} . \quad (3.6)$$

Just as in eq. (2.9), we have the algebra

$$[L_m^b, L_n^b] = (m-n) L_{m+n}^b , \quad (3.7)$$

aside from possible extra constants when $m+n=0$. Eq. (3.1b) allows us to define

$$L_n = L_n^a + L_n^b , \quad (3.8)$$

with the knowledge that these operators have the same algebra. (This is analogous to addition of angular momentum.) The L_n operators will enable us to demonstrate that our new model has much the same algebraic structure as the conventional one.

The next step is to formulate N -point functions. For this purpose we replace the τ -variable by $z = e^{i\tau}$ and define

$$V(k, z) = k \cdot H(z) V_0(k, z) . \quad (3.9)$$

The N -point function for N "pions" of mass m is then postulated to be

$$A_N(k_1, k_2, \dots, k_N) = \int d\mu_N(z) \prod_{i=1}^N \{ z^{-\frac{1}{2}} |z_i - z_{i+1}|^{\frac{1}{2} - m^2} \} \\ \times \langle 0 | V(k_1, z_1) V(k_2, z_2) \dots V(k_N, z_N) | 0 \rangle , \quad (3.10)$$

where the integration measure is given in eq. (2.12). The proof of cyclic symmetry proceeds as in the conventional model. First one proves Möbius invariance by considering the transformation of the vertex operator. Combining

$$[L_n^b, H^\mu(z)] = z^n \left(z \frac{d}{dz} + \frac{1}{2}n \right) H^\mu(z) \quad (3.11)$$

with eq. (2.15), one finds that for $k^2 = m^2$,

$$[L_n, V(k, z)] = z^n \left[z \frac{d}{dz} + n(\frac{1}{2} - m^2) \right] V(k, z) . \quad (3.12)$$

Thus, in analogy with eq. (2.14),

$$\Lambda(T) \frac{V(k, z)}{\sqrt{z}} \Lambda^{-1}(T) = (a - cz')^{1-2m^2} \frac{V(k, z')}{\sqrt{z'}}. \quad (3.13)$$

The extra factors $\prod_{i=1}^N (a - cz')^{1-2m^2}$ coming from eq. (3.13) are precisely cancelled by factors that arise in the transformation of the product appearing in eq. (3.10). The proof of cyclic symmetry is completed by means of the appropriate Möbius transformation (see sect. 2) and a study of the commutation properties of the vertices. Besides the phases of eq. (2.16) one obtains minus signs from the anticommutation of the $k \cdot H$. (The δ -functions of eq. (3.3) do not contribute to the integral.) The phase $(-1)^{N-1}$ that comes about in this way is in fact required in compensating phases arising from the transformation of $\left\{ \prod_{i=1}^N z_i^{-\frac{1}{2}} |z_i - z_{i+1}|^{\frac{1}{2} - m^2} \right\}$. Thus duality of the model is proved.

The amplitude A_N necessarily vanishes when N is odd, because the evaluation of $\langle 0 | k_1 \cdot H(z_1) \dots k_N H(z_N) | 0 \rangle$ requires that b -operators be paired. We may therefore define the G -parity operator by

$$G = (-1)^{m = \frac{1}{2} \sum_{m=1}^{\infty} b_{-m} \cdot b_m}, \quad (3.14)$$

and assert that our pions have negative G -parity.

By a construction analogous to that used to obtain eq. (2.18) we rewrite the N -point function in the form

$$A_N(k_1, k_2, \dots, k_N) = \langle 0 | k_1 \cdot b_{\frac{1}{2}} V(k_2) D V(k_3) D \dots D V(k_{N-1}) k_N \cdot b_{-\frac{1}{2}} | 0 \rangle, \quad (3.15)$$

with

$$D = \int_0^1 dx (1-x)^{-m^2 - \frac{1}{2}} x^{m^2 + L_0 - \frac{3}{2}}, \quad (3.16)$$

$$L_0 = R_a + R_b - p^2. \quad (3.17)$$

For the special value of the pion mass $m^2 = -\frac{1}{2}$, one has

$$D = (L_0 - 1)^{-1}. \quad (3.18)$$

In this case there are also Virasoro-type subsidiary conditions, $L_n \varphi = 0$, $n = 1, 2, \dots$. In complete analogy with eq. (2.23), eq. (3.12) enables one to show that for $m^2 = -\frac{1}{2}$,

$$(L_0 - L_n - 1) \frac{1}{L_0 - 1} V(k) = \frac{1}{L_0 + n - 1} V(k) (L_0 - L_n - 1). \quad (3.19)$$

Eq. (3.19), together with the fact that $L_0 - L_n - 1$ annihilates an on-shell pion (described by $k \cdot b_{-\frac{1}{2}} |0; k\rangle$ with $k^2 = -\frac{1}{2}$), proves the validity of the L_n subsidiary conditions. It is important to understand in this connection that

we only consider states of the Fock space to be physical when they can be made from multipion systems.

Now that the model has been formulated, let us begin an exploration of its implications by calculating the four-point function depicted in fig. 2.

$$\begin{aligned}
 A_4 &= \langle 0 | k_1 \cdot b_{\frac{1}{2}} V(k_2) D V(k_3) k_4 \cdot b_{-\frac{1}{2}} | 0 \rangle \\
 &= \int_0^1 dx x^{m^2-s-1} (1-x)^{m^2-t-1} \left\{ (k_1 \cdot k_2) (k_3 \cdot k_4) \sqrt{\frac{1-x}{x}} \right. \\
 &\quad \left. + (k_2 \cdot k_3) (k_1 \cdot k_4) \sqrt{\frac{x}{1-x}} - (k_1 \cdot k_3) (k_2 \cdot k_4) \sqrt{x(1-x)} \right\} \\
 &= (k_1 \cdot k_2) (k_3 \cdot k_4) B(-\alpha(s), 1-\alpha(t)) + (k_2 \cdot k_3) (k_1 \cdot k_4) B(1-\alpha(s), 1-\alpha(t)) \\
 &\quad - (k_1 \cdot k_3) (k_2 \cdot k_4) B(1-\alpha(s), 1-\alpha(t)) , \quad (3.20)
 \end{aligned}$$

where $\alpha(s) = s - m^2 + \frac{1}{2}$ is to be identified with the ρ -trajectory. If we require that there be no pole for $\alpha(s) = 0$, then it is necessary that

$$k_1 \cdot k_2 = \frac{1}{2}(s - 2m^2) = \frac{1}{2}\alpha(s) , \quad (3.21)$$

and hence that $m^2 = -\frac{1}{2}$. Remarkably enough, this is the same mass condition required for the Virasoro auges. When it is satisfied, eq. (3.20) may be simplified by some kinematics to

$$A_4 = -\frac{1}{4} \frac{\Gamma(1-\alpha(s)) \Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} . \quad (3.22)$$

This is the formula proposed for $\pi\pi$ scattering by Lovelace and Shapiro [14] some time ago. The factor of $\frac{1}{4}$ has no deep significance since we are omitting coupling constants and our pion state is not normalized:

$$\langle \pi | \pi \rangle = \langle 0 | k \cdot b_{\frac{1}{2}} k \cdot b_{-\frac{1}{2}} | 0 \rangle = -k^2 = \frac{1}{2} . \quad (3.23)$$

(Notice that it has positive norm only when it is a tachyon!) What we have succeeded in showing here is that for the special mass values $m_\pi^2 = -\frac{1}{2}$ and $m_\rho^2 = 0$, eq. (3.22) arises from a fully factorizable scheme.

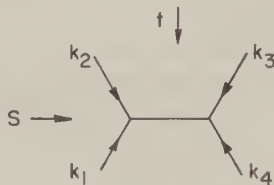


Fig. 2. Kinematics of $\pi\pi \rightarrow \pi\pi$.

4. BASIC CONJECTURE

At first sight the model we have constructed appears to have an unreasonable spectrum of states. The state described by the ground state of all

the oscillators, which we shall call G , is evidently a scalar meson with $M^2 = -1$ occurring for $\alpha_\rho(s) = 0$. Furthermore, with one $b_{-\frac{1}{2}}$ and $N-1$ α_{-1} excitations, it should be possible to make a state of spin N and $M^2 = N - \frac{1}{2}$. Such states would therefore lie on a trajectory one unit higher than the π -trajectory - hence we call it the A -(for ancestor) trajectory. Our basic conjecture is that G and the A -trajectory are spurious. If true, this would imply the existence of a rather profound gauge algebra in the model, going beyond the straightforward generalization of the Virasoro gauges. We call it a conjecture because we have not yet fully proved it, but we have no doubt about its truth.

In the previous section it was shown that for special mass values the G -pole does not occur in $\pi\pi$ scattering, which implies that G does not couple to two pions. Another easy calculation is a demonstration that the pion ancestor π_A (the first state on the A -trajectory) does not couple to three pions. It is trivially decoupled from an even number of pions by G -parity, of course. The π_A state is given by $\epsilon \cdot b_{-\frac{1}{2}} |0\rangle$ with $\epsilon \cdot k = 0$ and $k^2 = -\frac{1}{2}$. Therefore, the amplitude for $\pi_A \rightarrow 3\pi$, found by almost identical arithmetic to that of eq. (3.20), is

$$\begin{aligned} \langle 0 | \epsilon \cdot b_{\frac{1}{2}} V(k_2) D V(k_3) k_4 \cdot b_{-\frac{1}{2}} | 0 \rangle &= \epsilon \cdot k_2 (k_3 \cdot k_4) B(-\alpha(s), 1-\alpha(t)) \\ &+ \epsilon \cdot k_4 (k_2 \cdot k_3) B(1-\alpha(s), -\alpha(t)) - \epsilon \cdot k_3 (k_2 \cdot k_4) B(1-\alpha(s), 1-\alpha(t)) \\ &= -\frac{1}{2} \epsilon \cdot (k_2 + k_3 + k_4) \frac{\Gamma(1-\alpha(s)) \Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} = 0. \end{aligned}$$

The spurious character of the ground state and A -trajectory are not independent phenomena. The following arguments show that it is sufficient to know that either the state G or the state π_A is spurious to establish our conjecture. By definition, a spurious state is one that does not couple to any multiparticle system whose constituents are all physical. Therefore, a state which couples to a two-particle system consisting of a physical state and a spurious state must itself be spurious. We use this simple truth to establish the equivalence of π_A and G being spurious by showing that there is a non-vanishing $G - \pi - \pi_A$ vertex. Referring to fig. 3a, we have

$$\langle 0 | V(k_2) \epsilon \cdot b_{-\frac{1}{2}} | 0 \rangle = -\epsilon \cdot k_2. \quad (4.1)$$

But $\epsilon \cdot k_2 \neq 0$ is kinematically possible, so π_A is spurious if and only if G is. We can further show that if G is spurious the entire A -trajectory is spurious. In the notation of fig. 3b, it is sufficient to show that $V(k_2) | 0 \rangle$ contains the states of the A -trajectory. In particular, for $k^2 = N - \frac{1}{2}$ we can obtain a state of spin $N+1$ from the term

$$k_2 \cdot b_{-\frac{1}{2}} (k_2 \cdot \alpha_{-1})^N | 0 \rangle,$$

which arises in the expansion of $V(k_2) | 0 \rangle$. This term includes a spin $N+1$ part because k_2 contains a non-vanishing component orthogonal to k . Therefore, our conjecture would be established by proving G to be spurious. We have already seen that $\langle G | 2\pi \rangle = 0$. In the appendix we show that $\langle G | 4\pi \rangle = \langle G | 6\pi \rangle = 0$. While sufficient to convince us that G is spurious, the calculations of the appendix do not constitute a complete proof of that fact.

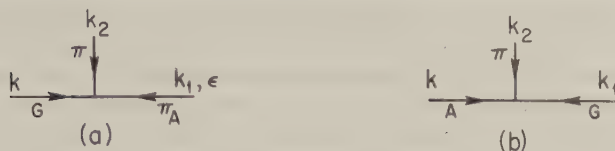


Fig. 3. a. The $G-\pi-\pi_A$ vertex. b. The $A-\pi-G$ vertex.

The nicest way to prove our conjecture would be by finding additional gauge operators (besides the L_n operators). So far we have been unsuccessful in this quest. In our opinion, the solution of this problem is required to obtain a complete understanding of what is going on in this model. It is quite remarkable that by adding superstructure to the conventional model, one can obtain a richer gauge group. It is *not* a structure that is present and previously overlooked in the conventional model. All the physical states evidently involve b -operators in their description, since otherwise one could isolate a subspace involving states built from α -states only.

It seems likely that one of the extra gauge operators, S , will commute with L_0 . The spuriousness of the ground state would then follow from the formula $\langle 0|S|0\rangle \neq 0$. Such a gauge operator is potentially very powerful in as much as there is no obvious limit to the number of spurious states that $S^\dagger$ can produce. This is in contrast to the action of L_{-n} which can only make as many spurious states of a given mass squared as the total number of states n units lower in mass squared. On the other hand, since $S^\dagger$ has the possibility of making so many spurious states it is conceivable that it makes all the spurious states, including those made by L_{-n} operators. Such a gauge operator we would choose to call a *supergauge*.

In our unsuccessful attempt to find the new gauge operators, we were led to construct some additional operators. We shall briefly mention them here since they might possibly prove useful to someone seeking the new gauges. These operators (quite similar to ones of Ramond's fermion model [23]) are

$$G_m = \sqrt{2} \langle e^{im\tau} P(\tau) \cdot H(\tau) \rangle = \sum_{n=-\infty}^{\infty} \alpha_n \cdot b_{m-n}, \quad m = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots, \quad (4.2)$$

and have the commutators and anticommutators

$$[L_m, G_n] = (\tfrac{1}{2}m - n) G_{m+n}, \quad (4.3)$$

$$\{G_m, G_n\} = 2L_{m+n}. \quad (4.4)$$

The G_m are certainly *not* the new gauges, but they might play a role in their construction.

5. EXPLORATION OF THE SPECTRUM

One of the first questions that arises in the consideration of quantum numbers is how to determine isospins. For this purpose the twist operator

is very useful because, as was pointed out in sect. 2, its eigenvalue gives the charge-conjugation quantum number for a physical state. Thus, as the G -parities of the states are easily determined by eq. (3.14), the isospins are immediately deduced. It is implicit in this reasoning that the only possibilities are $I = 0$ and $I = 1$ since we use the Chan-Paton procedure [21].

A natural guess for the twist operator is to replace eq. (2.25) by $(-1)^{R_a+R_b} e^{-L-1}$. This is almost, but not quite, correct. From eq. (3.11) it follows that

$$e^{-L-1} \frac{H_\mu(z)}{\sqrt{z}} e^{L-1} = \frac{H_\mu(z-1)}{\sqrt{z-1}}, \quad (5.1)$$

$$(-1)^{R_b} H_\mu(z-1) (-1)^{-R_b} = H_\mu(1-z). \quad (5.2)$$

Therefore, repeating eq. (2.26) for the b -portions of the vertices,

$$\begin{aligned} & (-1)^{R_b} e^{-L-1} k_1 \frac{H(z_1)}{\sqrt{z_1}} k_2 \frac{H(z_2)}{\sqrt{z_2}} \dots k_N \frac{H(z_N)}{\sqrt{z_N}} |0\rangle \\ &= (-1)^{\frac{1}{2}N} k_1 \frac{H(1-z_1)}{\sqrt{1-z_1}} k_2 \frac{H(1-z_2)}{\sqrt{1-z_2}} \dots k_N \frac{H(1-z_N)}{\sqrt{1-z_N}} |0\rangle \\ &= (-1)^{\frac{1}{2}N + \frac{1}{2}N(N-1)} k_N \frac{H(1-z_N)}{\sqrt{1-z_N}} k_{N-1} \frac{H(1-z_{N-1})}{\sqrt{1-z_{N-1}}} \dots k_1 \frac{H(1-z_1)}{\sqrt{1-z_1}} |0\rangle. \end{aligned} \quad (5.3)$$

In the last step we have used eq. (3.3), dropping the δ -functions since they do not contribute to the integrals. We thus conclude that the twist is given by

$$\Omega = (-1)^{R_a+R_b-\frac{1}{2}N^2} e^{-L-1}, \quad (5.4)$$

for an N -pion state. Eq. (5.4) is not yet in suitable operator form, however, since N has no operator representation. We therefore note that eq. (5.4) is equivalent to $(-1)^{R_a+R_b} e^{-L-1}$ for even- G states and $(-1)^{R_a+R_b-\frac{1}{2}} e^{-L-1}$ for odd- G states. Thus we may re-express eq. (5.4) in the form

$$\Omega = (-1)^{R_a+R_b-\frac{1}{4}(1-G)} e^{-L-1}. \quad (5.5)$$

We can now correlate isospins with masses. States with $M^2 = -\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \dots$ or $M^2 = 0, 2, 4, \dots$ must be isovector, whereas states with $M^2 = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \dots$ or $M^2 = 1, 3, 5, \dots$ must be isoscalar. In particular, the ρ and π are both isovector and may not be isospin doubled (i.e., degenerate with isoscalars). It also follows that there cannot be a σ degenerate with the ρ or a ρ' degenerate with f^0 . Furthermore, the amplitude for $\pi\pi \rightarrow \pi\pi$ cannot contain odd daughter trajectories. The reader may wonder whether we were forced to introduce isospin or whether the model is mathematically tenable without it.

The answer is that when the operator of eq. (3.14) is identified as G -parity the isospins are determined. If we insist that all states are isoscalar, then this operator must correspond to some other unphysical "modulo 2" rule.

There is a slight ambiguity in the parity assignments of states. The reason for this is that P and GP are equally good operators, which, in the absence of electromagnetism, fermions and strange mesons, cannot be distinguished. This fact leaves the freedom to switch the parity assignments of all the odd- G states. Therefore, we may *define* the π to be a pseudoscalar. All other parities are then determined.

We now embark on a detailed investigation of the spectrum. Not having the full gauge algebra in hand, another method for identifying spurious states is required. The method employed is to accept the validity of our conjecture that the ground state is spurious and then to deduce the existence of other spurious states therefrom. For example, we showed in sect. 4 that this implies that the A -trajectory is spurious. At the lowest mass squared, $M^2 = -1$, there is one spurious state, the ground state. At $M^2 = -\frac{1}{2}$ there is the physical pion given by $k \cdot b_{-\frac{1}{2}} |0\rangle$. This state has positive norm, but, as we mentioned before, this fact depends on its being a tachyon. The three spurious states, π_A , are also at $M^2 = -\frac{1}{2}$. (We count a state of spin S as $2S + 1$ states.)

At $M^2 = 0$ there are ten candidates for states given by $\alpha_{-1}^\mu |0\rangle$ and $b_{-\frac{1}{2}}^\mu b_{-\frac{1}{2}}^\nu |0\rangle$. One spurious state is given by

$$L_{-1} |0\rangle = \sqrt{2} k \cdot \alpha_{-1} |0\rangle. \quad (5.6)$$

This spurious scalar happens to have zero norm (because $k^2 = 0$). Additional spurious states can be found by studying the π - π_A system which can couple to spurious states only. Expanding out

$$V(k_2) \epsilon_1 \cdot b_{-\frac{1}{2}} |0\rangle, \quad (5.7)$$

with $\epsilon_1 \cdot k_1 = 0$, one finds that

$$\epsilon \cdot b_{-\frac{1}{2}} k \cdot b_{-\frac{1}{2}} |0\rangle, \quad (5.8)$$

$$\epsilon_{\mu\nu\lambda\sigma} k^\mu \epsilon^\nu b_{-\frac{1}{2}}^\lambda b_{-\frac{1}{2}}^\sigma |0\rangle, \quad (5.9)$$

are a spurious vector and axial vector, respectively. They also have zero norm. The only remaining candidate for a physical state is

$$|\rho\rangle = (\epsilon \cdot \alpha_{-1} + \beta \epsilon \cdot b_{-\frac{1}{2}} k \cdot b_{-\frac{1}{2}}) |0\rangle. \quad (5.10)$$

This state is orthogonal to the spurious vector for any choice of β . To determine β one expands out the physical 2π state

$$V(k_2) k_1 \cdot b_{-\frac{1}{2}} |0\rangle.$$

This yields the result $\beta = \sqrt{2}$. The ρ is easily seen to have positive norm.

At the next mass level, $M^2 = \frac{1}{2}$, the states must be odd- G and isoscalar. There are 24 candidates, which we now enumerate. First, we find the spurious states given by the standard gauges:

$$L_{-1} k \cdot b_{-\frac{1}{2}} |0\rangle = (\sqrt{2} k \cdot \alpha_{-1} k \cdot b_{-\frac{1}{2}} + k \cdot b_{-\frac{1}{2}}) |0\rangle, \quad (5.11a)$$

$$L_{-1}\epsilon \cdot b_{-\frac{1}{2}}|0\rangle = (\sqrt{2}k \cdot \alpha_{-1}\epsilon \cdot b_{-\frac{1}{2}} + \epsilon \cdot b_{-\frac{3}{2}})|0\rangle. \quad (5.11b)$$

Both of these spurious states have zero norm. Therefore, as explained by Del Giudice and Di Vecchia [24], it follows that the conjugate zero-norm states (obtained by $k \rightarrow -k$) are also spurious. Altogether this produces eight spurious zero-norm states. We obtain further spurious states by considering the two-particle system consisting of a π and a ground state. Expanding out $V(k_2)|0\rangle$, and doing some kinematics, one finds the following additional spurious states:

$$\epsilon_{\mu\nu} \alpha_{-1}^{\mu} b_{-\frac{1}{2}}^{\nu} |0\rangle, \quad (\text{spin } 2) \quad (5.12a)$$

$$\alpha_{-1} \cdot b_{-\frac{1}{2}} |0\rangle, \quad (5.12b)$$

$$[\epsilon \cdot b_{-\frac{3}{2}} - \sqrt{2}(\epsilon \cdot \alpha_{-1} k \cdot b_{-\frac{1}{2}} + k \cdot \alpha_{-1} \epsilon \cdot b_{-\frac{1}{2}})]|0\rangle. \quad (5.12c)$$

The occurrence of the spin-2 state in (5.12a) was expected since it lies on the A-trajectory. We have now found a total of 17 spurious states, so 7 remain to be investigated.

Six of the remaining states with $M^2 = \frac{1}{2}$ are

$$|\omega_1\rangle = \epsilon_{\mu\nu\lambda\sigma} \epsilon_{-1}^{\mu} \alpha_{-1}^{\nu} b_{-\frac{1}{2}}^{\lambda} k^{\sigma} |0\rangle \quad (5.13a)$$

$$|\omega_2\rangle = \frac{1}{6} \epsilon_{\mu\nu\lambda\sigma} \epsilon_{-1}^{\mu} b_{-\frac{1}{2}}^{\nu} b_{-\frac{1}{2}}^{\lambda} b_{-\frac{1}{2}}^{\sigma} |0\rangle. \quad (5.13b)$$

These states have the normalization $\langle\omega_1|\omega_1\rangle = 1$ and $\langle\omega_2|\omega_2\rangle = -1$ and have quantum numbers appropriate for the ω . A spurious combination is generated by considering the two-particle state consisting of a pion and a spurious axial ρ , i.e.,

$$V(k_2) \epsilon_{\mu\nu\lambda\sigma} k_1^{\mu} \epsilon_1^{\nu} b_{-\frac{1}{2}}^{\lambda} b_{-\frac{1}{2}}^{\sigma} |0\rangle.$$

In this way one finds the spurious state $|\omega_1\rangle - \sqrt{2}|\omega_2\rangle$, which has negative norm. A physical ω is obtained from a π plus a physical ρ , i.e., $V(k_2)(\epsilon_1 \cdot \alpha_{-1} + \sqrt{2}\epsilon_1 \cdot b_{-\frac{1}{2}} k_1 \cdot b_{-\frac{1}{2}})|0\rangle$. This calculation yields a state that is orthogonal to the spurious ω , namely

$$|\omega\rangle = \sqrt{2}|\omega_1\rangle - |\omega_2\rangle. \quad (5.14)$$

It is very pleasing to discover that this state has positive norm. The final remaining state with $M^2 = \frac{1}{2}$ is the scalar $\epsilon_{\mu\nu\lambda\sigma} k^{\mu} b_{-\frac{1}{2}}^{\nu} b_{-\frac{1}{2}}^{\lambda} b_{-\frac{1}{2}}^{\sigma} |0\rangle$. This state is spurious, because it also couples to a pion and the spurious axial ρ . Altogether, we have found that only 3 of the possible 24 states with $M^2 = \frac{1}{2}$ are physical. Evidently a very powerful gauge group is at work!

The amplitude for the process $\omega \rightarrow 3\pi$ can be calculated from $\langle\omega|V(k_3)DV(k_2)k_1 \cdot b_{-\frac{1}{2}}|0\rangle$. One easily finds

$$A(\omega \rightarrow 3\pi) = \frac{1}{3} \epsilon_{\mu\nu\lambda\sigma} \epsilon^{\mu} k_1^{\nu} k_2^{\lambda} k_3^{\sigma} B(1 - \alpha_{\rho}(s), 1 - \alpha_{\rho}(t)) . \quad (5.15)$$

Eq. (5.15) is recognized to be the original Veneziano formula [1].

To extend the discussion of the spectrum given above to higher masses would be a very tedious procedure. For example, in the case $M^2 = 1$ there are 55 states to study. We therefore choose to switch our attention now to the leading Regge trajectories. We have already seen that the A-trajectory is spurious. Another trajectory we can quickly dispose of is a parity double of the ρ - f^0 trajectory. Its operator description at spin N is

$$\left\{ \epsilon^{\mu 1}{}_{\nu\lambda\sigma} k^{\nu} b^{\lambda}{}_{-\frac{1}{2}} b^{\sigma}{}_{-\frac{1}{2}} \alpha^{\mu 2}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} |0\rangle . \quad (5.16)$$

The curly brackets in (5.16) imply the extraction of maximum spin. This means symmetrizing in μ indices, and removing all traces and divergences. This trajectory is spurious because it couples to the two-particle state consisting of a physical π plus a spurious π_A .

The remaining trajectories with unit intercept are the ρ and f^0 . In particular one wishes to determine their degeneracy. There are three candidates for spin N states on these trajectories:

$$\left\{ \alpha^{\mu 1}{}_{-1} \alpha^{\mu 2}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} |0\rangle , \quad (5.17a)$$

$$k \cdot b_{-\frac{1}{2}} \left\{ b^{\mu 1}{}_{-\frac{1}{2}} \alpha^{\mu 2}{}_{-1} \alpha^{\mu 3}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} |0\rangle , \quad (5.17b)$$

$$\left\{ b^{\mu 1}{}_{-\frac{1}{2}} b^{\mu 2}{}_{-\frac{3}{2}} \alpha^{\mu 3}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} |0\rangle . \quad (5.17c)$$

By considering the physical two-pion state $V(k_2) k_1 \cdot b_{-\frac{1}{2}} |0\rangle$, and doing some straightforward kinematics, one establishes that the following state is physical:

$$\left\{ \left[\alpha^{\mu 1}{}_{-1} \alpha^{\mu 2}{}_{-1} - \sqrt{2} k \cdot b_{-\frac{1}{2}} b^{\mu 1}{}_{-\frac{1}{2}} \alpha^{\mu 2}{}_{-1} + (N-1) b^{\mu 1}{}_{-\frac{1}{2}} b^{\mu 2}{}_{-\frac{1}{2}} \right] \alpha^{\mu 3}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} |0\rangle . \quad (5.18)$$

For $N = 1$, (5.18) represents the physical ρ , $(\alpha^{\mu 1}{}_{-1} - \sqrt{2} k \cdot b_{-\frac{1}{2}} b^{\mu 1}{}_{-\frac{1}{2}}) |0\rangle$. The norm of each of the states on the physical ρ - f^0 trajectory of (5.18) is positive. Consideration of the two-particle state consisting of a physical π and a spurious π_A gives rise to the following spurious state

$$\left\{ \left[(N-1) b^{\mu 1}{}_{-\frac{1}{2}} b^{\mu 2}{}_{-\frac{3}{2}} + \frac{1}{\sqrt{2}} k \cdot b_{-\frac{1}{2}} b^{\mu 1}{}_{-\frac{1}{2}} \alpha^{\mu 2}{}_{-1} \right] \alpha^{\mu 3}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} |0\rangle . \quad (5.19)$$

Similarly, the two-particle state consisting of a physical π and the spurious state $\epsilon \cdot \alpha_{-1} k \cdot b_{-\frac{1}{2}} |0\rangle$ gives the spurious state

$$\left\{ \left[\alpha^{\mu 1}{}_{-1} \alpha^{\mu 2}{}_{-1} - \sqrt{2} k \cdot b_{-\frac{1}{2}} b^{\mu 1}{}_{-\frac{1}{2}} \alpha^{\mu 2}{}_{-1} + (N-2) b^{\mu 1}{}_{-\frac{1}{2}} b^{\mu 2}{}_{-\frac{1}{2}} \right] \alpha^{\mu 3}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} |0\rangle . \quad (5.20)$$

The spurious states in (5.19) and (5.20) can be verified to be orthogonal to the physical state in (5.18) as must be the case, of course. We conclude that the physical ρ and f^0 trajectories are exchange-degenerate with one another, but are otherwise non-degenerate and contain positive norm states only.

Next let us consider the π -trajectory. It is expected to be exchange-degenerate with an odd-signature isoscalar trajectory, although we found no $J^P = 1^+$ state at $M^2 = \frac{1}{2}$. There are four candidates of spin N ,

$$k \cdot b_{-\frac{1}{2}} \left\{ \alpha_{-1}^{\mu 1} \alpha_{-1}^{\mu 2} \dots \alpha_{-1}^{\mu N} \right\} |0\rangle, \quad (5.21a)$$

$$k \cdot \alpha_{-1} \left\{ b_{-\frac{1}{2}}^{\mu 1} \alpha_{-1}^{\mu 2} \dots \alpha_{-1}^{\mu N} \right\} |0\rangle, \quad (5.21b)$$

$$\left\{ b_{-\frac{3}{2}}^{\mu 1} \alpha_{-1}^{\mu 2} \dots \alpha_{-1}^{\mu N} \right\} |0\rangle, \quad (5.21c)$$

$$\left\{ b_{-\frac{1}{2}}^{\mu 1} \alpha_{-2}^{\mu 2} \alpha_{-1}^{\mu 3} \dots \alpha_{-1}^{\mu N} \right\} |0\rangle. \quad (5.21d)$$

A spurious combination can be obtained through use of the gauge operator L_{-1} ,

$$\begin{aligned} L_{-1} \left\{ b_{-\frac{1}{2}}^{\mu 1} \alpha_{-1}^{\mu 2} \dots \alpha_{-1}^{\mu N} \right\} |0\rangle \\ = \left[\sqrt{2} k \cdot \alpha_{-1} \left\{ b_{-\frac{1}{2}}^{\mu 1} \alpha_{-1}^{\mu 2} \dots \alpha_{-1}^{\mu N} \right\} + \left\{ b_{-\frac{3}{2}}^{\mu 1} \alpha_{-1}^{\mu 2} \dots \alpha_{-1}^{\mu N} \right\} \right. \\ \left. + (N-1) \left\{ b_{-\frac{1}{2}}^{\mu 1} \alpha_{-2}^{\mu 2} \alpha_{-1}^{\mu 3} \dots \alpha_{-1}^{\mu N} \right\} \right] |0\rangle. \end{aligned} \quad (5.22)$$

The states of (5.22) have zero norm, and therefore the conjugate states obtained by replacing k by $-k$ are also spurious. A third spurious combination is contained in the two-particle state consisting of a π and a G , namely

$$\begin{aligned} \left[(N-\frac{1}{2}) \left\{ b_{-\frac{3}{2}}^{\mu 1} \alpha_{-1}^{\mu 2} \dots \alpha_{-1}^{\mu N} \right\} + 2(N-\frac{1}{2})(N-1) \left\{ b_{-\frac{1}{2}}^{\mu 1} \alpha_{-2}^{\mu 2} \alpha_{-1}^{\mu 3} \dots \alpha_{-1}^{\mu N} \right\} \right. \\ \left. - \frac{1}{\sqrt{2}} k \cdot b_{-\frac{1}{2}} \left\{ \alpha_{-1}^{\mu 1} \dots \alpha_{-1}^{\mu N} \right\} - \frac{N}{\sqrt{2}} k \cdot \alpha_{-1} \left\{ b_{-\frac{1}{2}}^{\mu 1} \alpha_{-1}^{\mu 2} \dots \alpha_{-1}^{\mu N} \right\} \right] |0\rangle. \end{aligned} \quad (5.23)$$

The remaining candidate for the physical trajectory is the trajectory orthogonal to the three spurious ones,

$$\left[\sqrt{2} k \cdot b_{-\frac{1}{2}} \left\{ \alpha_{-1}^{\mu 1} \dots \alpha_{-1}^{\mu N} \right\} + N \left\{ b_{-\frac{1}{2}}^{\mu 1} \alpha_{-2}^{\mu 2} \alpha_{-1}^{\mu 3} \dots \alpha_{-1}^{\mu N} \right\} - 2N \left\{ b_{-\frac{3}{2}}^{\mu 1} \alpha_{-1}^{\mu 2} \dots \alpha_{-1}^{\mu N} \right\} \right] |0\rangle. \quad (5.24)$$

The spin- N state in (5.24) has positive norm. To see that it is in fact physical one can consider the states coupling to a π plus a physical ρ . When this

is done, one finds (5.24) multiplied by $N-1$ plus an irrelevant admixture of a zero-norm spurious state. One therefore deduces that (5.24) is physical except for $N=1$! This exception means that the Regge-residue functions have a zero at this point. The absence of this state is fortunate from a phenomenological point of view since there is no good candidate for a $J^P = 1^+$, $IG = 0^-$ state with mass around 1 GeV. On the other hand, there is a candidate for the pion recurrence at mass 1640 MeV.

Next we consider the ω - A_2 trajectory, which happens to be degenerate with the π -trajectory in our model but is distinguished by its parity. There are three candidates for this trajectory:

$$\left\{ \epsilon^{\mu 1}{}_{\nu \lambda \sigma} b^{\nu}{}_{-\frac{1}{2}} b^{\lambda}{}_{-\frac{1}{2}} b^{\sigma}{}_{-\frac{1}{2}} \alpha^{\mu 2}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} |0\rangle, \quad (5.25a)$$

$$\left\{ \epsilon^{\mu 1}{}_{\nu \lambda \sigma} k^{\nu} \alpha^{\lambda}{}_{-1} b^{\sigma}{}_{-\frac{1}{2}} \alpha^{\mu 2}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} |0\rangle, \quad (5.25b)$$

$$\left\{ \epsilon^{\mu 1}{}_{\nu \lambda \sigma} k^{\nu} b^{\lambda}{}_{-\frac{1}{2}} b^{\sigma}{}_{-\frac{1}{2}} b^{\mu 2}{}_{-\frac{3}{2}} \alpha^{\mu 3}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} |0\rangle. \quad (5.25c)$$

A spurious combination can be obtained from the state consisting of a π and a spurious ρ , and a different spurious combination can be found from a π and a spurious f^0 . Thus there is at most one physical trajectory. Physical states are generated by a π plus the physical ρ . One finds the result

$$\begin{aligned} & \left[(1-2N) \left\{ \epsilon^{\mu 1}{}_{\nu \lambda \sigma} b^{\nu}{}_{-\frac{1}{2}} b^{\lambda}{}_{-\frac{1}{2}} b^{\sigma}{}_{-\frac{1}{2}} \alpha^{\mu 2}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} \right. \\ & \quad + 6\sqrt{2} \left\{ \epsilon^{\mu 1}{}_{\nu \lambda \sigma} k^{\nu} \alpha^{\lambda}{}_{-1} b^{\sigma}{}_{-\frac{1}{2}} \alpha^{\mu 2}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} \\ & \quad \left. + 3\sqrt{2}(N-1) \left\{ \epsilon^{\mu 1}{}_{\nu \lambda \sigma} k^{\nu} b^{\lambda}{}_{-\frac{1}{2}} b^{\sigma}{}_{-\frac{1}{2}} b^{\mu 2}{}_{-\frac{3}{2}} \alpha^{\mu 3}{}_{-1} \dots \alpha^{\mu N}{}_{-1} \right\} \right] |0\rangle. \quad (5.26) \end{aligned}$$

These states also have positive norm. The ω and A_2 trajectories are exchange degenerate and quite satisfactory in all respects.

On the basis of the physical trajectories that have been found so far, we can formulate a remarkable empirical rule. Some of the trajectories of our model are approximately one-half unit higher than their experimental values, whereas others are very close to their experimental locations. Furthermore these two classes of trajectories appear to be distinguished by their parities. The precise statement is that those states with normal-parity couplings to pions, i.e., $P = G(-1)^S$, where S is the spin of the state, occur $\frac{1}{2}$ unit too high. Equivalently, the mass is given by $M^2 = M_{\text{expt}}^2 - \frac{1}{2}$. On the other hand, abnormal-parity states, those for which $P = -G(-1)^S$, have $M^2 = M_{\text{expt}}^2$. One may conjecture that this rule will give the model predictive power for states that may not have been seen experimentally. The relationship between the model and the real world is quite intriguing.

We conclude this section with a few comments about other states. Two states are conspicuous by their absence. They are the σ -meson, which is

generally believed to be a broad resonance degenerate with the ρ , and the A_1 meson. The experimental evidence in each case is still generally considered to be inconclusive. Whether our model constitutes an argument against them is debatable.

There are two other physical states we shall comment on. One is a scalar meson degenerate with the f^0 . Its existence is already evident from eq. (3.22). We have not examined its degeneracy or norm. Another state of considerable interest is a pseudoscalar at the same mass. Its quantum numbers are those of the η , but the empirical mass rule strongly suggests that an identification be made with the $\eta'(960)$. There are two candidates for this state

$$|\eta'_1\rangle = \frac{1}{\sqrt{12}} \epsilon_{\mu\nu\lambda\sigma} k^\mu \alpha^\nu_{-1} b^\lambda_{-\frac{1}{2}} b^\sigma_{-\frac{1}{2}} |0\rangle, \quad (5.27a)$$

$$|\eta'_2\rangle = \frac{1}{24} \epsilon_{\mu\nu\lambda\sigma} b^\mu_{-\frac{1}{2}} b^\nu_{-\frac{1}{2}} b^\lambda_{-\frac{1}{2}} b^\sigma_{-\frac{1}{2}} |0\rangle. \quad (5.27b)$$

These states have the normalization $\langle\eta'_1|\eta'_1\rangle = 1$ and $\langle\eta'_2|\eta'_2\rangle = -1$. By constructing the coupling to a π and a member of the A_2 trajectory it is possible to show that the physical combination is

$$|\eta'\rangle = \sqrt{3}|\eta'_1\rangle + \sqrt{2}|\eta'_2\rangle, \quad (5.28)$$

which has positive norm. The orthogonal combination is spurious.

6. THE SIX-PION AMPLITUDE

In order to acquire a more concrete understanding of the workings of the model, it is helpful to work out explicitly the six-pion amplitude. For the configuration shown in fig. 4 we have the formula

$$A_6 = \langle 0 | k_1 \cdot b_{\frac{1}{2}} V(k_2) DV(k_3) DV(k_4) DV(k_5) k_6 \cdot b_{-\frac{1}{2}} | 0 \rangle. \quad (6.1)$$

The evaluation of this expression gives rise to 15 terms corresponding to the different ways of pairing the six sets of b -oscillators. After a straightforward, but somewhat tedious, calculation one finds

$$\begin{aligned} A_6 = & \int \frac{du_{12} du_{13} du_{14}}{(1 - u_{12} u_{13})(1 - u_{13} u_{14})} u_{12}^{-\alpha_\rho(s_{12})} u_{13}^{-\alpha_\pi(s_{13})-1} u_{23}^{-\alpha_\rho(s_{23})} \\ & \times u_{34}^{-\alpha_\rho(s_{34})} u_{45}^{-\alpha_\rho(s_{45})} u_{24}^{-\alpha_\pi(s_{24})-1} u_{35}^{-\alpha_\pi(s_{35})-1} u_{14}^{-\alpha_\rho(s_{14})} \\ & \times u_{25}^{-\alpha_\rho(s_{25})} Y(k, u), \end{aligned} \quad (6.2)$$

where the u_{ij} variables correspond to the notation of Goebel and Sakita [2] and

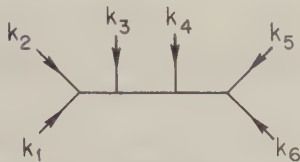


Fig. 4. Kinematics of the six-pion amplitude.

$$\begin{aligned}
 Y(k, u) = & \left[k_1 \cdot k_2 k_3 \cdot k_4 k_5 \cdot k_6 \frac{1}{u_{12} u_{14} u_{34}} + k_1 \cdot k_6 k_4 \cdot k_5 k_2 \cdot k_3 \frac{1}{u_{23} u_{45} u_{25}} \right] \\
 & + \left[k_1 \cdot k_2 k_3 \cdot k_6 k_4 \cdot k_5 \frac{u_{24}}{u_{12} u_{45}} + k_1 \cdot k_4 k_2 \cdot k_3 k_5 \cdot k_6 \frac{u_{35}}{u_{23} u_{14}} \right. \\
 & + k_1 \cdot k_6 k_2 \cdot k_5 k_3 \cdot k_4 \frac{u_{13}}{u_{34} u_{25}} \left. \right] + \left[k_1 \cdot k_4 k_2 \cdot k_6 k_3 \cdot k_6 u_{13} u_{24} \right. \\
 & + k_1 \cdot k_3 k_2 \cdot k_5 k_4 \cdot k_6 u_{24} u_{35} + k_1 \cdot k_5 k_2 \cdot k_4 k_3 \cdot k_6 u_{13} u_{35} \left. \right] \\
 & - \left[k_1 \cdot k_2 k_3 \cdot k_5 k_4 \cdot k_6 \frac{u_{24}}{u_{12}} + k_1 \cdot k_5 k_2 \cdot k_3 k_4 \cdot k_6 \frac{u_{35}}{u_{23}} \right] \\
 & + k_1 \cdot k_5 k_2 \cdot k_6 k_3 \cdot k_4 \frac{u_{13}}{u_{34}} + k_1 \cdot k_3 k_2 \cdot k_6 k_4 \cdot k_5 \frac{u_{24}}{u_{45}} \\
 & + k_1 \cdot k_3 k_2 \cdot k_4 k_5 \cdot k_6 \frac{u_{35}}{u_{14}} + k_1 \cdot k_6 k_3 \cdot k_5 k_2 \cdot k_4 \frac{u_{13}}{u_{25}} \left. \right] \\
 & - \left[k_1 \cdot k_4 k_2 \cdot k_5 k_3 \cdot k_6 u_{13} u_{24} u_{35} \right]. \quad (6.3)
 \end{aligned}$$

In this formula we have grouped terms together which transform into one another under cyclic permutations of the momenta. Thus the cyclic symmetry of A_6 is manifest.

Fortunately there is a trick that allows a significant simplification in the formula for A_6 . We know that the pion ancestor, π_A , does not couple to five pions. (This is in fact proved since we show in the appendix that G does not couple to six pions.) On the other hand, the amplitude for $\pi_A \rightarrow 5\pi$ is obtained by replacing the momentum k_1 in eq. (6.3) by a polarization vector ϵ . The only way the various terms can then add to zero is if the coefficients of $\epsilon \cdot k_2, \epsilon \cdot k_3, \dots, \epsilon \cdot k_6$ are all equal so that the sum is proportional to $\epsilon \cdot (k_2 + k_3 + k_4 + k_5 + k_6) = 0$. Although it is really not necessary to do so, we have checked that the five terms in question are, in fact, equal. Therefore, returning to the six-pion amplitude, we may replace $Y(k, u)$ by the coefficient of $k_1 \cdot k_6$, say. This amounts to replacing Y by

$$\tilde{Y}(k, u) = k_4 \cdot k_5 k_2 \cdot k_3 \frac{1}{u_{23} u_{45} u_{25}} + k_2 \cdot k_5 k_3 \cdot k_4 \frac{u_{13}}{u_{34} u_{25}} + k_3 \cdot k_5 k_2 \cdot k_4 \frac{u_{13}}{u_{25}}. \quad (6.4)$$

The verification of cyclic symmetry when A_6 is expressed in terms of $\tilde{Y}$ is not manifest. One method is to discover judicious integrations by parts.

The amplitude for $\rho \rightarrow 4\pi$ is obtained by taking the residue of A_6 at $\alpha_\rho(s_{14}) = 1$. One finds (see fig. 5)

$$\begin{aligned} A(\rho \rightarrow 4\pi) = & \int \frac{du_{12} du_{13}}{1 - u_{12} u_{13}} u_{12}^{-\alpha_\rho(s_{12})} u_{13}^{-\alpha_\pi(s_{13})-1} u_{23}^{-\alpha_\rho(s_{23})} \\ & \times u_{34}^{-\alpha_\rho(s_{34})} u_{24}^{-\alpha_\pi(s_{24})-1} \left[\epsilon \cdot k_4 k_2 \cdot k_3 \frac{1}{u_{23}} \right. \\ & \left. + \epsilon \cdot k_2 k_3 \cdot k_4 \frac{u_{13}}{u_{34}} + \epsilon \cdot k_3 k_2 \cdot k_4 u_{13} \right]. \quad (6.5) \end{aligned}$$

The amplitude for $\pi\rho \rightarrow \pi\rho$ (see fig. 6a) then follows from taking the residue of eq. (6.5) at $\alpha_\rho(s_{12}) = 1$:

$$\begin{aligned} A(\pi\rho \rightarrow \pi\rho) = & -\frac{1}{2} \epsilon_1 \cdot \epsilon_2 (1 - \alpha_\pi(s) - \alpha_\rho(t)) B(1 - \alpha_\pi(s), 1 - \alpha_\rho(t)) \\ & + \epsilon_1 \cdot k_3 \epsilon_2 \cdot k_4 B(-\alpha_\pi(s), 1 - \alpha_\rho(t)) \\ & - \epsilon_1 \cdot k_4 \epsilon_2 \cdot k_3 B(1 - \alpha_\pi(s), 1 - \alpha_\rho(t)). \quad (6.6) \end{aligned}$$

An alternative method of deriving eq. (6.6) is by directly evaluating

$$\langle 0 | (\epsilon_1 \cdot \alpha_1 + \sqrt{2} \epsilon_1 \cdot b_{\frac{1}{2}} k_1 \cdot b_{\frac{1}{2}}) V(k_3) D V(k_4) (\epsilon_2 \cdot \alpha_{-1} + \sqrt{2} \epsilon_2 \cdot b_{-\frac{1}{2}} k_2 \cdot b_{-\frac{1}{2}}) | 0 \rangle. \quad (6.7)$$

Eq. (6.6) clearly contains a π -pole in the s -channel and a ρ -pole in the t -channel. Somewhat less obvious is the nature of the pole at $\alpha_\pi(s) = 1$. According to the analysis of sect. 5 the only possible singularity at this mass is the ω . The residue of eq. (6.6) at $\alpha_\pi(s) = 1$ is

$$\beta(t) = \frac{1}{2} \alpha_\rho(t) \epsilon_1 \cdot \epsilon_2 + \alpha_\rho(t) \epsilon_1 \cdot k_3 \epsilon_2 \cdot k_4 - \epsilon_1 \cdot k_4 \epsilon_2 \cdot k_3. \quad (6.8)$$

On the other hand, by a straightforward calculation one can show that for $s = \frac{1}{2}$,

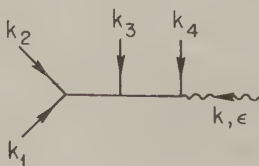


Fig. 5. Kinematics for $\rho \rightarrow 4\pi$.

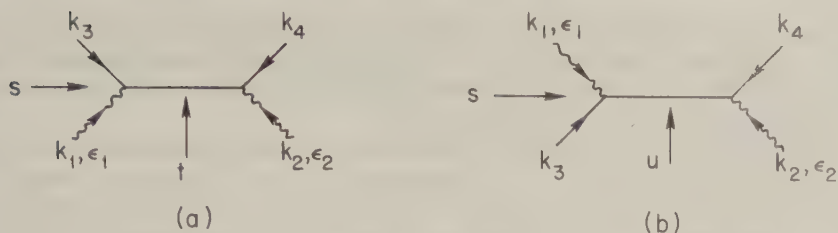


Fig. 6. a. Kinematics for $\pi\rho \rightarrow \pi\pi$. b. Kinematics for $\pi\rho \rightarrow \rho\pi$.

$$\epsilon_{\mu\nu\lambda\sigma} \epsilon_1^{\mu} k_1^{\nu} k_3^{\lambda} \epsilon_{\mu'\nu'\lambda'} \epsilon_2^{\sigma} k_2^{\mu'} k_4^{\nu'} k_4^{\lambda'} = \beta(t). \quad (6.9)$$

Thus the residue is completely accounted for by the ω -resonance, without any admixture of 1^+ , 0^- , etc.

The amplitude shown in fig. 6b is obtained by factorizing eq. (6.5) at the pole $\alpha_\rho(s_{23}) = 1$:

$$\begin{aligned} A(\pi\rho \rightarrow \rho\pi) = & -\frac{1}{2} \epsilon_1 \cdot \epsilon_2 (1 - \alpha_\pi(s) - \alpha_\pi(u)) B(1 - \alpha_\pi(s), 1 - \alpha_\pi(u)) \\ & + \epsilon_1 \cdot k_3 \epsilon_2 \cdot k_4 B(-\alpha_\pi(s), 1 - \alpha_\pi(u)) \\ & + \epsilon_1 \cdot k_4 \epsilon_2 \cdot k_3 B(1 - \alpha_\pi(s), -\alpha_\pi(u)). \end{aligned} \quad (6.10)$$

In contrast to eq. (6.6), this expression contains odd- G poles dual to themselves. It also contains only an ω -pole at $\alpha_\pi(s) = 1$, as is easily shown by the same sort of analysis as before.

More complicated amplitudes, such as $\rho\rho \rightarrow \rho\rho$, could be obtained by methods similar to those in this section. However, a more efficient route to such results would probably begin by deriving the general operator expression for the scattering of N excited states. Such a formula has been found in the conventional model [25]. In any case, the main purpose of this section was to verify some of the assertions of the previous sections for a non-trivial example.

7. CONCLUSION

The dual model presented here bears some resemblance to the real world of mesons, while still retaining the good features of the conventional model. Its spectrum is approximately the same as predicted by a naive model in which mesons are made from $q\bar{q}$ states involving non-strange quarks only. Such a rule is required for multipion states in a dual quark model, and it accounts for finding ω and f^0 trajectories and not φ and f' . Stretching the observed mixing angle a bit, it could also account for finding an η' and not an η . In any case, just as with the empirical mass rule of sect. 5, these statements are only a description of what the model seems to be doing and are not based on any particularly deep understanding. A deeper under-

standing could be provided by a model for underlying quarks, for example. Such a model might be useful for seeing how to make a sensible extension to SU(3). This is an important problem, because a naive SU(3) extension of the present model gives a bad spectrum.

The most important feature of our model is its gauge algebra. It is responsible for the appearance of an enormous number of spurious states. Specifically, it was shown in sect. 5 that only 3 of the 12 leading trajectories are physical. The new gauges furthermore provide the key to obtaining a spurious state at the point where the $\rho - f^0$ trajectory crosses zero. An important problem remaining to be solved is the explicit construction of the new gauge operators. Such formulas would both allow for a simplification of the algebra in sect. 5 (and an elimination of the appendix!) and provide a basis for a much better understanding of some beautiful mathematics that is at present hidden from view.

Just as the Virasoro gauge algebra played an important role in motivating the construction of this model, so - we hope - will the new algebra implied by this model contribute to the construction of a still better one.

APPENDIX

In this appendix we show that the ground state G decouples from any state containing 4 or 6 pions. Using the notations of fig. 7a, we can write the amplitude for G going into four pions as:

$$A(G \rightarrow 4\pi) = 4 \int dx_1 dx_2 x_1^{-\alpha_\rho(12)-1} x_2^{-\alpha_\pi(13)-1} (1-x_1)^{-2k_2 \cdot k_3} (1-x_2)^{-2k_3 \cdot k_4} \\ \times (1-x_1 x_2)^{-2k_2 \cdot k_4} \left[\frac{(k_1 \cdot k_2)(k_3 \cdot k_4)}{x_1(1-x_2)} + \frac{(k_1 \cdot k_4)(k_2 \cdot k_3)}{1-x_1} - \frac{(k_1 \cdot k_3)(k_2 \cdot k_4)}{1-x_1 x_2} \right]. \quad (A.1)$$

We then expand $(1-x_1 x_2)^{-2k_2 \cdot k_4}$ in a power series and integrate over x_1 and x_2 to get

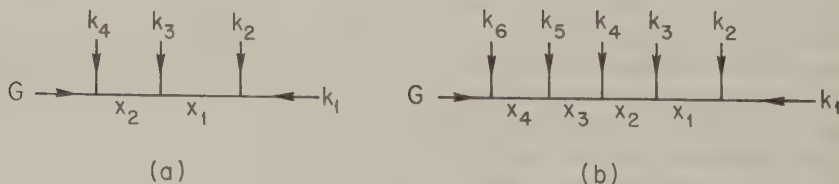


Fig. 7. a. The $G \rightarrow 4\pi$ amplitude. b. The $G \rightarrow 6\pi$ amplitude.

$$\begin{aligned}
A_4 = & 4(k_1 \cdot k_2)(k_3 \cdot k_4) \sum_{n=0}^{\infty} \frac{\Gamma(n+2k_2 \cdot k_4)}{n! \Gamma(2k_2 \cdot k_4)} B[n - \alpha_{\pi}(13), -2k_3 \cdot k_4] \\
& \times B[n - \alpha_{\rho}(12) - 1, 1 - 2k_2 \cdot k_3] + 4(k_1 \cdot k_4)(k_2 \cdot k_3) \sum_{n=0}^{\infty} \frac{\Gamma(n+2k_2 \cdot k_4)}{n! \Gamma(2k_2 \cdot k_4)} \\
& \times B[n - \alpha_{\pi}(13), 1 - 2k_3 \cdot k_4] B[n - \alpha_{\rho}(12), -2k_2 \cdot k_3] - 2(k_1 \cdot k_3) \\
& \times \sum_{n=0}^{\infty} \frac{\Gamma(n+2k_2 \cdot k_4 + 1)}{n! \Gamma(2k_2 \cdot k_4)} B[n - \alpha_{\pi}(13), 1 - 2k_3 \cdot k_4] B[n - \alpha_{\rho}(12), 1 - 2k_2 \cdot k_3] .
\end{aligned} \tag{A.2}$$

Next we express the B -functions in terms of Γ -functions and use the following kinematics:

$$2k_1 \cdot k_2 = [\alpha_{\rho}(12) - n] + [n] ,$$

$$2k_1 \cdot k_4 = [-\alpha_{\pi}(13) - 2k_3 \cdot k_4 + n] - [n + 2k_2 \cdot k_4] ,$$

$$2k_1 \cdot k_3 = [-\alpha_{\rho}(12) - 2k_2 \cdot k_3 + n + 1] + [\alpha_{\pi}(13) - n] .$$

It is then trivial to check that the terms in square brackets in these three equations give rise to contributions which cancel by pairs.

In the amplitude A_6 for G going into 6 pions shown in fig. 7b, there are 15 terms multiplying the ordinary Bardakçi-Ruegg integrand [2]. These fifteen terms are

$$\begin{aligned}
T_1 = 8 \frac{(k_1 \cdot k_2)(k_3 \cdot k_4)(k_5 \cdot k_6)}{(1-x_2)(1-x_4)} , & \quad T_8 = 8 \frac{(k_1 \cdot k_6)(k_4 \cdot k_5)(k_2 \cdot k_3)x_1x_3}{(1-x_1)(1-x_3)} , \\
T_2 = 8 \frac{(k_1 \cdot k_3)(k_2 \cdot k_5)(k_4 \cdot k_6)x_1x_3}{(1-x_1x_2x_3)(1-x_3x_4)} , & \quad T_9 = -8 \frac{(k_1 \cdot k_2)(k_3 \cdot k_5)(k_4 \cdot k_6)x_3}{(1-x_2x_3)(1-x_3x_4)} , \\
T_3 = 8 \frac{(k_1 \cdot k_2)(k_3 \cdot k_6)(k_4 \cdot k_5)x_3}{(1-x_3)(1-x_2x_3x_4)} , & \quad T_{10} = -8 \frac{(k_1 \cdot k_3)(k_2 \cdot k_6)(k_4 \cdot k_5)x_1x_3}{(1-x_1x_2x_3x_4)(1-x_3)} , \\
T_4 = 8 \frac{(k_1 \cdot k_4)(k_2 \cdot k_3)(k_5 \cdot k_6)x_1}{(1-x_1)(1-x_4)} , & \quad T_{11} = -8 \frac{(k_1 \cdot k_4)(k_3 \cdot k_6)(k_2 \cdot k_5)x_1x_2x_3}{(1-x_2x_3x_4)(1-x_1x_2x_3)} , \\
T_5 = 8 \frac{(k_1 \cdot k_4)(k_2 \cdot k_6)(k_3 \cdot k_5)x_1x_2x_3}{(1-x_2x_3)(1-x_1x_2x_3x_4)} , & \quad T_{12} = -8 \frac{(k_1 \cdot k_3)(k_2 \cdot k_4)(k_5 \cdot k_6)x_1}{(1-x_1x_2)(1-x_4)} , \\
T_6 = 8 \frac{(k_1 \cdot k_5)(k_2 \cdot k_4)(k_3 \cdot k_6)x_1x_2x_3}{(1-x_1x_2)(1-x_2x_3x_4)} , & \quad T_{13} = -8 \frac{(k_1 \cdot k_5)(k_2 \cdot k_6)(k_3 \cdot k_4)x_1x_2x_3}{(1-x_1x_2x_3x_4)(1-x_2)} , \\
T_7 = 8 \frac{(k_1 \cdot k_6)(k_2 \cdot k_5)(k_3 \cdot k_4)x_1x_2x_3}{(1-x_1x_2x_3)(1-x_2)} , & \quad T_{14} = -8 \frac{(k_1 \cdot k_5)(k_2 \cdot k_3)(k_4 \cdot k_6)x_1x_3}{(1-x_1)(1-x_3x_4)} .
\end{aligned}$$

$$T_{15} = -8 \frac{(k_1 \cdot k_6) (k_3 \cdot k_5) (k_2 \cdot k_4) x_1 x_2 x_3}{(1 - x_2 x_3) (1 - x_1 x_2)}.$$

We then expand in power series all the factors of the Bardakci-Ruegg integrand which involve more than one x . This means expanding the following product:

$$(1 - x_1 x_2)^{-2k_2 \cdot k_4} (1 - x_2 x_3)^{-2k_3 \cdot k_5} (1 - x_3 x_4)^{-2k_4 \cdot k_6} (1 - x_1 x_2 x_3)^{-2k_2 \cdot k_5} \\ \times (1 - x_2 x_3 x_4)^{-2k_3 \cdot k_6} (1 - x_1 x_2 x_3 x_4)^{-2k_2 \cdot k_6}.$$

Let us call n_{24} , n_{35} , n_{46} , n_{25} , n_{36} , n_{26} the integers (running from zero to infinity) involved in the expansion of each factor. We then integrate explicitly over x_1 , x_2 , x_3 , x_4 each term of the series, and use the following kinematics:

$$\text{in } T_1, T_3, T_9: \quad 2k_1 \cdot k_2 = [\alpha_\rho(12) - n_{24} - n_{25} - n_{26}] + [n_{24}] + [n_{25}] + [n_{26}];$$

$$\text{in } T_2, T_{10}, T_{12}: \quad 2k_1 \cdot k_3 = [-\alpha_\rho(12) - 2k_2 \cdot k_3 + n_{24} + n_{25} + n_{26} + 1] \\ + [n_{35}] + [n_{36}] + [\alpha_\pi(13) - n_{24} - n_{25} - n_{26} - n_{35} - n_{36}];$$

$$\text{in } T_4: \quad 2k_1 \cdot k_4 = [-\alpha_\pi(13) - 2k_3 \cdot k_4 + n_{24} + n_{35} + n_{25} + n_{36} + n_{26}] \\ + [n_{46}] - [n_{24} + 2k_2 \cdot k_4] \\ + [\alpha_\rho(14) - n_{46} - n_{35} - n_{25} - n_{36} - n_{26}];$$

$$\text{in } T_5, T_{11}: \quad 2k_1 \cdot k_4 = [-\alpha_\pi(13) - 2k_3 \cdot k_4 + n_{24} + n_{35} + n_{25} + n_{36} + n_{26} + 1] \\ + [n_{46}] - [n_{24} + 2k_2 \cdot k_4] \\ + [\alpha_\rho(14) - n_{46} - n_{35} - n_{25} - n_{36} - n_{26} - 1];$$

$$\text{in } T_6, T_{13}, T_{14}: \quad 2k_1 \cdot k_5 = [-\alpha_\rho(14) + n_{35} + n_{46} + n_{25} + n_{36} + n_{26} + 1 - 2k_4 \cdot k_5] \\ - [n_{25} + 2k_2 \cdot k_5] - [n_{35} + 2k_3 \cdot k_5] \\ + [\alpha_\pi(15) - n_{46} - n_{36} - n_{26}];$$

$$\text{in } T_7, T_8, T_{15}: \quad 2k_1 \cdot k_6 = [-\alpha_\pi(15) + n_{46} + n_{36} + n_{26} - 2k_5 \cdot k_6] \\ - [n_{26} + 2k_2 \cdot k_6] - [n_{36} + 2k_3 \cdot k_6] \\ - [n_{46} + 2k_4 \cdot k_6].$$

It is left as an exercise to the reader to show that all the terms in brackets above, when taken with the appropriate series, cancel one another by pairs.

It seems that such a pedestrian proof of the decoupling of the ground state cannot be generalized in practice to more than 6 pions, owing to the enormous complexity of the formulas.

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REFORMULATION OF THE DUAL PION MODEL *

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The dual pion model is reformulated in another Fock space. The basic conjecture about the decoupling of the ground state is proved, and all the on-shell gauge operators are found.

In a recent paper [1] a new dual-resonance model of pions was formulated. Notable among its properties are 1) The absence of a spin-zero tachyon associated with the passage of the ρ trajectory through zero. 2) Half-unit spacing between even- G and odd- G trajectories. 3) The occurrence of states having abnormal-parity couplings to pions. 4) The apparent absence of ghosts. 5) The nondegeneracy of the ρ - f^0 , ω - A_2 and π trajectories. Actually in ref. [1] it was only proved that these properties follow from the conjecture that the ground state decouples from any number of pions. The conjecture was shown to be correct for six or fewer pions. In the present work simple algebraic methods are used to prove this conjecture for the general case. As an outgrowth of the techniques employed we are naturally led to reformulate the model in another Fock space ($\mathcal{F}_2$). The new space has the advantage of containing fewer spurious states and allowing the spectrum of physical particle states to be studied more easily. It turns out that in $\mathcal{F}_2$ the gauge group consists only of the G_m 's and L_n 's defined in ref. [1] and below.

We begin by recalling some of the notation introduced in ref. [1]. The standard harmonic oscillator and momentum operators are replaced by

$$\alpha_n^\mu = -i\sqrt{n}a_n^\mu, \quad \alpha_{-n}^\mu = i\sqrt{n}a_n^{\dagger\mu}, \quad \alpha_0^\mu = \sqrt{2}p^\mu.$$

In addition, we introduce anticommuting four-vector operators b_m^μ for $m = \pm 1/2, \pm 3/2, \dots$, defining $b_{-m}^\mu = b_m^{\dagger\mu}$. The algebra of the α 's and b 's is then postulated to be

$$[\alpha_m^\mu, \alpha_n^\nu] = -mg^{\mu\nu}\delta_{m+n,0}, \quad \{b_m^\mu, b_n^\nu\} = -g^{\mu\nu}\delta_{m+n,0}, \quad [\alpha_m^\mu, b_n^\nu] = 0.$$

Of particular importance in this paper are the operators

$$G_m = \sum_{n=-\infty}^{\infty} \alpha_n \cdot b_{m-n}, \quad m = \pm 1/2, \pm 3/2 \quad (1)$$

whose anticommutators give the Virasoro operators [2]

$$\{G_m, G_n\} = 2L_{m+n}. \quad (2)$$

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In our notation, the vertex of the conventional model is

$$V_0(k) = \exp(ik \cdot x) \exp\left(\sqrt{2} k \cdot \sum_{n=1}^{\infty} \frac{\alpha_{-n}}{n}\right) \exp\left(-\sqrt{2} k \cdot \sum_{n=1}^{\infty} \frac{\alpha_n}{n}\right),$$

while for the dual-pion model the vertex is

$$V(k) = k \cdot H V_0(k)$$

with

$$H^\mu = \sum_{m=-\infty}^{\infty} b_m^\mu \quad \text{and} \quad k^2 = -1/2.$$

The key formula for this paper is

$$-\sqrt{2} \{G_m, V(k)\} = (L_0 + m - 1/2) V_0(k) - V_0(k) (L_0 - 1/2), \quad (3)$$

which shall be used repeatedly to derive a number of interesting relations. To prove eq. (3) one first notes that

$$[G_m, V_0(k)] = -\sqrt{2} V(k) \quad (4)$$

is an easy consequence of the definitions given above. The proof is then completed by anticommuting both sides of eq. (4) with G_m ,

$$-\sqrt{2} \{G_m, V(k)\} = [L_{2m}, V_0(k)] = L_{2m} V_0(k) + V_0(k) (L_0 - L_{2m} - 1/2) - V_0(k) (L_0 - 1/2) = \\ (L_0 + m - 1/2) V_0(k) - V_0(k) (L_0 - 1/2).$$

We can now prove that the ground state does not couple to any number of pions. A helpful step is to rewrite the pion state $\sqrt{2} k \cdot b_{-1/2} |0\rangle$ in the equivalent form $G_{-1/2} |0\rangle$. Once this is done, the amplitude for the coupling of a ground state to $N-1$ pions is given by

$$\langle 0 | V(k_2) \frac{1}{L_0 - 1} V(k_3) \dots \frac{1}{L_0 - 1} V(k_{N-1}) G_{-1/2} | 0 \rangle \quad (5)$$

We then use eq. (3) for $m = -1/2$ to work $G_{-1/2}$ to the left. Since $(L_0 - 1/2) |0\rangle = 0$ and $\langle 0 | G_{-1/2} = 0$ the only terms remaining to be considered are of the form

$$\langle 0 | V(k_2) \frac{1}{L_0 - 1} \dots V(k_M) V_0(k_{M+1}) \frac{1}{L_0 - 1/2} \dots V(k_{N-1}) | 0 \rangle$$

and

$$\langle 0 | V(k_2) \frac{1}{L_0 - 1} \dots V_0(k_M) V(k_{M+1}) \frac{1}{L_0 - 1/2} \dots V(k_{N-1}) | 0 \rangle$$

In each case one of the propagators has been cancelled. Such terms are zero because the effect of removing the propagator between V and V_0 is equivalent to the insertion of a δ function in the integral representation of the propagator forcing the equality of adjacent Koba-Nielsen variables [3]. Another way of showing that they vanish is to recall that expressions such as these can be expanded in terms of the poles of any channel along the multiperipheral chain, in particular the channel of the missing propagator. Such an expansion trivially gives zero, because there are no poles in that channel. The basic conjecture of ref. [1] is therefore proved.

Let us now consider the N -pion amplitude, applying the same techniques as in the preceding paragraph.

$$A_N = \langle 0 | G_{1/2} V(k_2) \frac{1}{L_0 - 1} V(k_3) \dots \frac{1}{L_0 - 1} V(k_{N-1}) G_{-1/2} | 0 \rangle = \quad (6)$$

$$\langle 0 | G_{1/2} G_{-1/2} V(k_2) \frac{1}{L_0 - 1/2} V(k_3) \dots \frac{1}{L_0 - 1/2} V(k_{N-1}) | 0 \rangle = \langle 0 | V(k_2) \frac{1}{L_0 - 1/2} V(k_3) \dots \frac{1}{L_0 - 1/2} V(k_{N-1}) | 0 \rangle$$

Eq. (6) allows a reinterpretation of the model in terms of a new Fock space $\mathcal{F}_2$, the original Fock space being called $\mathcal{F}_1$. In $\mathcal{F}_2$ the ground state with $k^2 = -1/2$ represents the pion and the propagator is $1/(L_0 - 1/2)$. The formula for the pion emission vertex is the same as in $\mathcal{F}_1$. While it is clearly natural to use the same operator symbols for both $\mathcal{F}_1$ and $\mathcal{F}_2$ it is important to realize that they really are quite distinct. One advantage of $\mathcal{F}_2$ is that neither a state corresponding to the passage of the ρ trajectory through zero nor states of an 'ancestor trajectory' ever appear even as spurious states. Another useful property of $\mathcal{F}_2$ is that the G_m 's, $m = 1/2, 3/2, \dots$, are on-shell gauge operators. This is proved by using the same techniques as above to show that

$$\langle \psi, L_0 = 1/2 - m | G_m V(k_2) \frac{1}{L_0 - 1/2} V(k_3) \dots \frac{1}{L_0 - 1/2} V(k_{N-1}) | 0 \rangle = 0.$$

The L 's are also on-shell gauge operators of $\mathcal{F}_2$ because $L_{2m} = G_m^2$.

The cyclic symmetry of the N -pion amplitude was proved in ref. [1]. That proof was carried out in the $\mathcal{F}_1$ formulation, and it is therefore interesting to give a new proof directly in $\mathcal{F}_2$. This is achieved by first using the methods of ref. [1] to show (by a somewhat lengthy calculation) that

$$\begin{aligned} \langle 0; k_1 | V(k_2) \frac{1}{L_0 - 1/2} V(k_3) \dots \frac{1}{L_0 - 1/2} V(k_{N-1}) | 0; k_N \rangle = \\ \langle 0; k_N | V_0(k_1) \frac{1}{L_0 - 1} V(k_2) \dots \frac{1}{L_0 - 1} V(k_{N-2}) k_{N-1} \cdot b_{-1/2} | 0; k_{N-1} \rangle. \end{aligned}$$

Then, just as in previous arguments, we rewrite this as

$$\begin{aligned} \frac{1}{\sqrt{2}} \langle 0; k_N | V_0(k_1) \frac{1}{L_0 - 1} V(k_2) \dots \frac{1}{L_0 - 1} V(k_{N-2}) G_{-1/2} | 0; k_{N-1} \rangle = \\ \frac{1}{\sqrt{2}} \langle 0; k_N | V_0(k_1) G_{-1/2} \frac{1}{L_0 - 1/2} V(k_2) \dots \frac{1}{L_0 - 1/2} V(k_{N-2}) | 0; k_{N-1} \rangle = \\ \langle 0; k_N | V(k_1) \frac{1}{L_0 - 1/2} V(k_2) \dots \frac{1}{L_0 - 1/2} V(k_{N-2}) | 0; k_{N-1} \rangle, \end{aligned}$$

where the last step uses eq. (4).

Next let us explore the relationship between physical states of $\mathcal{F}_1$ and $\mathcal{F}_2$. We let $|\phi_1\rangle$ and $|\phi_1'\rangle$ denote physical on-mass-shell states of $\mathcal{F}_1$ and $|\phi_2\rangle$ and $|\phi_2'\rangle$ the corresponding states of $\mathcal{F}_2$. The connections to be proved are

$$|\phi_1\rangle = G_{-1/2} |\phi_2\rangle \quad (7a)$$

and

$$|\phi_2\rangle = G_{1/2} |\phi_1\rangle. \quad (7b)$$

Since $G_{1/2} |\phi_2\rangle = (L_0 - 1/2) |\phi_2\rangle = 0$,

$$|\phi_2\rangle = (2L_0 - G_{-1/2} G_{1/2}) |\phi_2\rangle = G_{1/2} G_{-1/2} |\phi_2\rangle. \quad (8)$$

Table 1

Intercept	$\mathcal{F}_2$ description of spin N state	Physical spin values
1 ($\rho - f^0$)	$\{b_{-1/2}^{\mu_1} \alpha_{-1}^{\mu_2} \dots \alpha_{-1}^{\mu_N}\} 0 \rangle$	1, 2, 3, ...
1/2 ($\omega - A_2$)	$\{\epsilon_{\nu\lambda\sigma}^{\mu_1} k^\nu b_{-1/2}^\lambda b_{-1/2}^\sigma \alpha_{-1}^{\mu_2} \dots \alpha_{-1}^{\mu_N}\} 0 \rangle$	1, 2, 3, ...
1/2 ($\pi - ?$)	$\{[\alpha_{-1}^{\mu_1} \alpha_{-2}^{\mu_2} + N b_{-1/2}^{\mu_1} b_{-3/2}^{\mu_2}] \alpha_{-1}^{\mu_3} \dots \alpha_{-1}^{\mu_N}\} 0 \rangle$	0, 2, 3, ...
0	$\{\epsilon_{\nu\lambda\sigma}^{\mu_1} [2k^\nu \alpha_{-1}^\lambda b_{-1/2}^\sigma \alpha_{-1}^{\mu_2} - (N+1) k^\nu b_{-1/2}^\lambda b_{-1/2}^\sigma b_{-3/2}^{\mu_2}] \alpha_{-1}^{\mu_3} \dots \alpha_{-1}^{\mu_N}\} 0 \rangle$	2, 3, 4, ...

Table 2

M^2	IG	JP	Name	$\mathcal{F}_2$ Description *
$-1/2$	1^-	0^-	π	$ 0\rangle$
0	1^+	1^-	ρ	$b_{-1/2}^\mu 0\rangle$
$1/2$	0^-	1^-	ω	$\epsilon_{\nu\lambda\sigma}^\mu k^\nu b_{-1/2}^\lambda b_{-1/2}^\sigma 0\rangle$
1	0^+	2^+	f^0	$b_{-1/2}^{\mu_1} \alpha_{-1}^{\mu_2} 0\rangle$
1	1^-	1^-	π	$\epsilon_{\mu\nu\lambda\sigma} k^\mu b_{-1/2}^\nu b_{-1/2}^\lambda b_{-1/2}^\sigma 0\rangle$
1	0^+	0^+	σ	$(b_{-1/2} \cdot \alpha_{-1} - k \cdot b_{-1/2} k \cdot \alpha_{-1}) 0\rangle$
$3/2$	1^-	2^+	A_2	$\alpha_{-1}^{\mu_1} \epsilon_{\nu\lambda\sigma}^\mu k^\nu b_{-1/2}^\lambda b_{-1/2}^\sigma 0\rangle$
$3/2$	1^-	2^-	π_A	$(b_{-1/2}^{\mu_1} b_{-3/2}^{\mu_2} + 1/2 \alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2}) 0\rangle$
$3/2$	1^-	1^+	A_1'	$(\sqrt{2} k \cdot b_{-3/2} + 2 \alpha_{-1} \cdot b_{-1/2} - 2 k \cdot b_{-1/2} k \cdot \alpha_{-1}) b_{-1/2}^\mu 0\rangle$
$3/2$	1^-	0^-	π'	$(2 k \cdot b_{-3/2} k \cdot b_{-1/2} - k \cdot \alpha_{-1} k \cdot \alpha_{-1} + 3 b_{-1/2} \cdot b_{-3/2} + 3/2 \alpha_{-1} \cdot \alpha_{-1}) 0\rangle$

* In certain cases the description is unique only within a zero-norm state.

This enables us to make the correspondence between $\mathcal{F}_2$ and $\mathcal{F}_1$ amplitudes needed to establish eq. (7a) as follows:

$$\begin{aligned}
 \langle \phi_2' | V(k_2) \frac{1}{L_0 - 1/2} V(k_3) \dots \frac{1}{L_0 - 1/2} V(k_{N-1}) | \phi_2 \rangle &= \\
 \langle \phi_2' | V(k_2) \frac{1}{L_0 - 1/2} V(k_3) \dots \frac{1}{L_0 - 1/2} V(k_{N-1}) G_{1/2} G_{-1/2} | \phi_2 \rangle &= \\
 \langle \phi_2' | G_{1/2} V(k_2) \frac{1}{L_0 - 1/2} V(k_3) \dots \frac{1}{L_0 - 1/2} V(k_{N-1}) G_{-1/2} | \phi_2 \rangle &= \langle \phi_1' | V(k_2) \frac{1}{L_0 - 1/2} V(k_3) \dots \frac{1}{L_0 - 1/2} V(k_{N-1}) | \phi_1 \rangle
 \end{aligned}$$

Eq. (7b) is an immediate consequence of eqs. (7a) and (8).

Another interesting relation is

$$G_{-1/2} G_{1/2} | \phi \rangle = G_{-1/2} | \phi_2 \rangle = | \phi_1 \rangle \quad (9)$$

This formula implies that

$$S = G_{-1/2} G_{1/2} \quad (10)$$

is an on-shell gauge operator of $\mathcal{F}_1$. This operator is of special interest since it implies that the ground state and ancestor trajectory are spurious. Other on-shell gauge operators of $\mathcal{F}_1$ are $G_m G_{1/2}$ for $m = 1, 2, 3, 2, \dots$ and L_n for $n = 1, 2, \dots$.

The spectrum of physical states is easier to work out in $\mathcal{F}_2$ than in $\mathcal{F}_1$. There are two basic reasons for this. The first is that a given state requires $1/2$ unit less excitation for its operator description in $\mathcal{F}_2$ than in $\mathcal{F}_1$. The second is that the gauge operators are simpler in $\mathcal{F}_2$. In table 1 we tabulate all the physical states on trajectories with intercept $1, 1/2$, and 0 . In table 2 we tabulate all the states with $M^2 \approx 3/2$. Each of the states tabulated has a positive norm. It therefore seems quite likely that the model has no ghosts.

In conclusion we would like to point out that the results of this paper suggest a set of rules for inventing other interesting models.

- 1) Start with any model that possesses conformal invariance under L_n 's.
- 2) Introduce additional oscillators which give rise to the same algebra and construct the sum algebra.

$$L_n = L_n^{\text{old}} + L_n^{\text{new}}$$

- 3) Find the 'square root' representation of L_n , call it $G_{n/2}$, and define the new vertex to be $[G_{1/2}, V_0(k)]$.

Models of this type will have many of the properties, including additional Ward identities, that the present model possesses.

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Quark Model of Dual Pions*

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Interacting pseudoscalar pions are incorporated into Ramond's model of free dual fermions. By considering the emission of $N - 1$ pions and factorizing in the quark-antiquark channel, we recover the same N -pion amplitudes as were proposed in a previous paper.

Ramond¹ has recently proposed a model of free dual fermions related to the Dirac equation by a correspondence principle² analogous to one relating the conventional dual-resonance model to the Klein-Gordon equation. This fermion model possesses an infinite set of Ward identities that probably provides for the cancellation of all ghosts. Another recent development was the discovery of a dual model of pions³ having a number of realistic features not shared by the conventional dual-resonance model. Subsequently, the algebraic properties responsible for the successes of this model (including the apparent absence of ghosts) have been obtained.⁴

In this paper we show that there is a deep connection between the fermion and pion models. Specifically, we construct the amplitude for emitting $N - 1$ pions from a fermion line [Fig. 1(a)]. Requiring the gauge algebra of the fermion sector to hold in the presence of interaction imposes the condition $m_n^2 = -\frac{1}{2}$, the same condition required for the gauges in the meson sector. By factorizing at the first pole in the quark-antiquark channel [Fig. 1(b)], we obtain the same N -pion amplitude as in Ref. 3.

Let us first review the algebra of Ramond's fermion model. In addition to the usual harmonic-oscillator operators⁵ satisfying

$$[\alpha_m^\mu, \alpha_n^\nu] = -m g^{\mu\nu} \delta_{m, -n},$$

Ramond introduces anticommuting operators satisfying

$$\{d_m^\mu, d_n^\nu\} = -g^{\mu\nu} \delta_{m, -n}$$

and

$$[d_m^\mu, \alpha_n^\nu] = 0,$$

where m and n are integers, $d_{-m}^\mu = d_m^{\mu\dagger}$, and $d_0^\mu = -(i/\sqrt{2})\gamma_5\gamma^\mu$. Then, introducing

$$P^\mu(\tau) = (\tfrac{1}{2})^{1/2} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in\tau}$$

and

$$\Gamma^\mu(\tau) = i\sqrt{2} \gamma_5 \sum_{n=-\infty}^{\infty} d_n^\mu e^{-in\tau},$$

one finds that the operators

$$L_n = -\langle e^{in\tau}; P^2(\tau); \rangle + \tfrac{1}{4}i \left\langle e^{in\tau}; \Gamma(\tau) \cdot \frac{d}{d\tau} \Gamma(\tau); \right\rangle$$

satisfy the Virasoro algebra⁶

$$[L_m, L_n] = (m - n)L_{m+n}.$$

The wave equation for a fermion state is

$$(F_0 - m)|\psi\rangle = 0,$$

where m is the mass of the spin- $\frac{1}{2}$ ground-state fermion and

$$F_n = \langle e^{in\tau} \Gamma(\tau) \cdot P(\tau) \rangle.$$

Ramond's subsidiary ghost-eliminating conditions are

$$F_n |\psi\rangle = 0, \quad n = 1, 2, 3, \dots \quad (1)$$

or, equivalently,

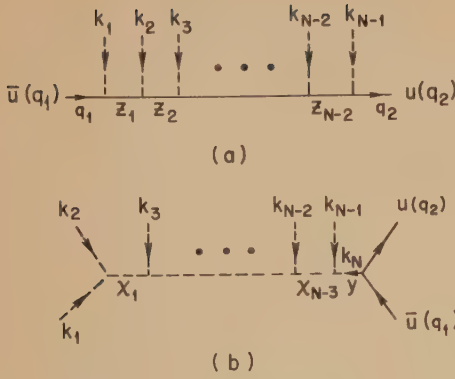


FIG. 1. (a) Emission of $N-1$ pions from a quark.
(b) Factorization in the quark-antiquark channel.

$$L_n |\psi\rangle = 0, \quad n=1, 2, 3, \dots \quad (2)$$

The remaining algebra of the F_n 's and L_n 's is

$$\begin{aligned} X_n \frac{1}{F_0 - m} &= [F_0 - F_n - m + c_n(L_n - L_0 - m^2)] \frac{1}{F_0 - m} \\ &= \frac{1}{F_0 - (m^2 + n)^{1/2}} [F_n - F_0 - m + c_n(L_n - L_0 - m^2 - \frac{1}{2}n)], \end{aligned}$$

with

$$c_n = (2/n)[(m^2 + n)^{1/2} + m].$$

Therefore, since the ordinary vertex operator

$$V_0(k) = e^{ik \cdot x} \exp\left(\sqrt{2} k \cdot \sum_{n=1}^{\infty} \frac{\alpha_n}{n}\right) \exp\left(-\sqrt{2} k \cdot \sum_{n=1}^{\infty} \frac{\alpha_n}{n}\right)$$

commutes with $F_0 - F_n$ and shifts $L_0 - L_n$ by nk^2 , Eq. (5) can be satisfied by multiplying V_0 by an expression that anticommutes with $F_0 - F_n$. The simplest choice with this property⁷ is

$$\Gamma_5 = \gamma_5 (-1)^{\sum_{n=1}^{\infty} (a_n^\dagger \cdot a_n)}.$$

Equation (5) is then satisfied if the emitted meson has $m^2 = -\frac{1}{2}$. It is interesting to note that the fermion model requires the emitted meson to be pseudoscalar, whereas in Ref. 3 the pion parity was undetermined.

The amplitude for the process shown in Fig. 1(a) is

$$B_{N-1} = \bar{u}(q_1) \left\langle 0 \left| \Gamma_5 V_0(k_1) \frac{1}{F_0 - m} \Gamma_5 V_0(k_2) \frac{1}{F_0 - m} \cdots \frac{1}{F_0 - m} \Gamma_5 V_0(k_{N-1}) \right| 0 \right\rangle u(q_2).$$

Rewriting the propagators in the form $-(F_0 + m)/(L_0 + m^2)$ and pushing the Γ_5 's to the left gives

$$B_{N-1} = (-1)^{(N/2)+1} \bar{u}(q_1) \gamma_5 \left\langle 0 \left| V_0(k_1) \frac{F_0 + m}{L_0 + m^2} V_0(k_2) \frac{F_0 - m}{L_0 + m^2} \cdots \frac{F_0 - m}{L_0 + m^2} V_0(k_{N-1}) \right| 0 \right\rangle u(q_2).$$

Next we take the $F_0 \pm m$ factors to the right through the vertices using

$$[F_0, V_0(k)] = -V_0(k) k \cdot \Gamma,$$

and dropping terms with cancelled propagators.⁴ This procedure results in the formula

$$B_{N-1} = (-1)^{(N/2)+1} \bar{u}(q_1) \gamma_5 \left\langle 0 \left| V_0(k_1) \frac{1}{L_0 + m^2} k_2 \cdot \Gamma V_0(k_2) \frac{1}{L_0 + m^2} \cdots \frac{1}{L_0 + m^2} k_{N-1} \cdot \Gamma V_0(k_{N-1}) \right| 0 \right\rangle u(q_2). \quad (6)$$

The advantage of expressing B_{N-1} in the form of Eq. (6) is that after substituting the standard integral representation

$$[L_m, F_n] = (\frac{1}{2}m - n)F_{m+n}, \quad (3)$$

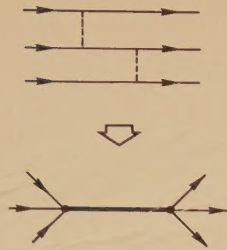
$$\{F_m, F_n\} = -2L_{m+n}. \quad (4)$$

The key requirement for introducing interaction into Ramond's model is that it be compatible with the subsidiary conditions (1) and (2). The propagator for a fermion line will of course be $1/(F_0 - m)$. Therefore, in analogy with the conventional model, we seek off-shell gauge operators X_n and a meson-emission vertex $V(k)$ such that

$$X_n \frac{1}{F_0 - m} V(k) = \frac{1}{F_0 - (m^2 + n)^{1/2}} V(k) X_n. \quad (5)$$

If X_n annihilates the on-shell fermion ground state, then Eq. (5) implies that it annihilates an arbitrary tree state. Using Eqs. (3) and (4) one can show that

FIG. 2. Possible scheme for forming baryons from three quarks.



$$\frac{1}{L_0 + m^2} = \int_0^1 z^{L_0 + m^2 - 1} dz,$$

the algebra of α and d modes separates. The contribution of the α modes to the integrand is essentially of the Bardakci-Ruegg form.⁸ The algebra

$$B_{N-1} \underset{(q_1 - q_2)^2 \rightarrow m_\pi^2}{\sim} \bar{u}(q_1) \gamma_5 u(q_2) \frac{1}{(q_1 - q_2)^2 - m_\pi^2} A_N(k_1, k_2, \dots, k_N = q_1 - q_2),$$

where, in the notation of Ref. 4,

$$A_N(k_1, k_2, \dots, k_N) = \left\langle 0 \left| k_2 \cdot H V_0(k_2) \frac{1}{L_0 - \frac{1}{2}} k_3 \cdot H V_0(k_3) \cdots \frac{1}{L_0 - \frac{1}{2}} k_{N-1} \cdot H V_0(k_{N-1}) \right| 0 \right\rangle.$$

This proves our assertion that Ramond's fermion model leads to the dual-pion model in the meson sector.

In conclusion, we wish to point out that the connection between the amplitude depicted in Fig. 1(b) and the amplitudes of the dual-pion model has been demonstrated only at the first pole in the $q\bar{q}$ channel. It should, however, be possible to make an operator generalization to describe the coupling of an arbitrary meson state to the quark. Once this is done, one could attempt, for example, to construct baryons out of three quarks as shown in Fig. 2. In such a channel one would expect the pole

of d modes, on the other hand, is similar to that of the b modes occurring in the computation of the N -pion amplitude A_N in the formulation of Ref. 4.

The last remark is the key to showing the connection between A_N and B_{N-1} . First, one makes the standard change of variables to go from the configuration of Fig. 1(a) to the one of Fig. 1(b), namely,

$$x_i = \frac{1 - z_1 z_2 \cdots z_i}{1 - z_1 z_2 \cdots z_{i+1}}, \quad i = 1, 2, \dots, N-3$$

$$y = 1 - z_1 z_2 \cdots z_{N-2}.$$

Then factorizing in the quark-antiquark channel near the first pole $[(q_1 - q_2)^2 = m_\pi^2 = -\frac{1}{2}]$, one finds that

positions to depend on the quark mass in contrast to what we have found for the $q\bar{q}$ channel. The quarks could be only mathematical since one would have a consistent scheme by considering just the states with zero triality. Clearly, many interesting problems remain to be studied.

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$$\alpha_n^\mu = -i\sqrt{n} a_n^\mu, \quad \alpha_{-n}^\mu = i\sqrt{n} a_n^{\mu\dagger}, \quad \alpha_0^\mu = \sqrt{2} p^\mu.$$

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